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ON POLYNOMIAL EXTENSIONS OF RINGS

MICHIO YOSHIDA

Let A be a commutative ring with unit element, and let $A[x]$ be a ring of polynomials in an indeterminate x with coefficients in A. There are a number of well-known properties which A shares with $A[x]$. We shall state one of them in the following.

THEOREM. *If A is an integrally closed integral domain, then so also is $A[x]$.*

In an earlier volume of this journal, Messrs. Butts, Hall and Mann **(1)** gave a proof of the theorem. The purpose of the present note is to give a simpler elementary proof and another valuation-theoretic one.

First proof. Let K be the quotient field of A. At first we assume that K is algebraically closed. If $f(x) \in K(x)$ is integral over $A[x]$, then $f(x) \in K[x]$, since $K[x]$ is integrally closed. Since $f(x)$ satisfies a monic equation with coefficients in $A[x]$, for any element ξ in A, $f(\xi)$ is integral over A, that is, $f(\xi) \in A$. Set

$$f(x) = a_0 + a_1 x + \ldots + a_m x^m \qquad (a_i \in K),$$

and take $m+1$ distinct elements ξ_j $(0 \leqslant j \leqslant m)$ in A. Then

$$a_0 + a_1\xi_j + \ldots + a_m\xi_j^m = \zeta_j, \quad \text{with } \zeta_j \in A \ (0 \leqslant j \leqslant m).$$

We solve these equations with respect to $a_0, a_1, \ldots, a_m$ and obtain

$$a_i = \frac{\eta_i}{D}, \text{ where } \eta_i \in A, D = \prod_{i<j} (\xi_i - \xi_j).$$

Here we notice that ξ_j $(0 \leqslant j \leqslant m)$ can be chosen such that $D = 1$, since any monic equation of the form $(x-\xi_1)(x-\xi_2)\ldots(x-\xi_k) = 1$, with $\xi_i \in A$, has a solution in A.

We now turn to the case in which K is not algebraically closed. Let $\bar{K}$ be the algebraic closure of K and let $\bar{A}$ be the integral closure of A in $\bar{K}$. Then the integral closure of $A[x]$ in $K(x)$ is contained in $K(x) \cap \bar{A}[x] = A[x]$.

Second proof. We first recall the following fact. Let ν be a valuation of a field K, then ν can be extended to a valuation $\bar{\nu}$ of $K(x)$ by setting, for any polynomial

$$f(x) = a_0 + a_1 x + \ldots + a_m x^m$$

in $K[x]$,

$$\bar{\nu}[f(x)] = \min_{0 \leqslant i \leqslant m} \nu(a_i).$$

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Since A is integrally closed, A can be represented as an intersection of a set of valuation rings of K: $A = \cap R_\iota$. Denote by $\tilde{R}_\iota$ the valuation ring of $K(x)$, which is uniquely deduced from R_ι in the manner described just above. Then we have obviously

$$A[x] = \cap \tilde{R}_\iota \cap K[x].$$

Since $\tilde{R}_\iota$ and $K[x]$ are integrally closed, so is $A[x]$.

Remark. An element a of K is said to be *almost integral over* A, if there exists an element $b \neq 0$ of A such that $ba^n \in A$ for all n. If any element of K which is almost integral over A is contained in A, A is said to be *fully integrally closed*. We note that in our theorem the phrase "integrally closed" can be replaced by "fully integrally closed." This can be proved as follows.

Let $f(x) \in K(x)$ be almost integral over $A[x]$, so that $f(x) \in K[x]$. We shall show that $f(x) \in A[x]$, by induction with respect to the degree of $f(x)$. By the definition of "almost integral," there exists a non-zero polynomial $g(x)$ in $A[x]$, such that

$$g(x)\, f(x)^\nu \in A[x]$$

for any positive integer ν. Let b, a be the leading coefficients of $g(x)$ and $f(x)$ respectively, and put $f(x) = ax^m + f_1(x)$. Then $ba^\nu \in A$, hence $a \in A$; consequently

$$g(x)\, f_1(x)^\nu = g(x)[f(x) - ax^m]^\nu \in A[x],$$

whence by the induction assumption, $f_1(x) \in A[x]$, hence $f(x) \in A[x]$.

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Hiroshima University

A THEOREM ON ZARISKI RINGS

MICHIO YOSHIDA

Given a Noetherian ring A with unit element and an ideal $\mathfrak{m}$ of A such that

$$\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0),$$

we may topologize A by adopting $\{\mathfrak{m}^n; n = 1, 2, \ldots\}$ as a fundamental system of neighborhoods of 0. This topologized ring is usually referred to as an $\mathfrak{m}$-adic ring, and is called a Zariski ring if its ideals are all closed. An $\mathfrak{m}$-adic ring is a Zariski ring if and only if $\mathfrak{m}$ is contained in its Jacobson radical (i.e., the intersection of all its maximal ideals).

Let A be an $\mathfrak{m}$-adic Zariski ring, and let us denote its completion by $\hat{A}$. Let $\mathfrak{a}, \mathfrak{b}$ be arbitrary ideals of A; then a theorem due to P. Samuel (**4**, p. 158, Theorem 4; **1**, p. 54, Theorem 1) says:

$$(\mathfrak{a} \cap \mathfrak{b})\hat{A} = \mathfrak{a}\hat{A} \cap \mathfrak{b}\hat{A}.$$

In this note, we shall derive two consequences of this theorem. We shall first state the following theorem, which seems not to be found in the literature, at least not in its full generality, though it is known in some special cases and has been found useful.

THEOREM 1. *Let $\mathfrak{a}$ be an arbitrary ideal of A, then if $\mathfrak{a}\hat{A}$ is generated by r elements, so also is $\mathfrak{a}$. In particular, if $\mathfrak{a}\hat{A}$ is principal, so is $\mathfrak{a}$.*

Proof. Suppose that $\mathfrak{a}\hat{A} = (\hat{a}_1, \ldots, \hat{a}_r)$, and take integer ρ (**5**, p. 47, Lemma 3) such that $\mathfrak{m}\mathfrak{a} \supseteq \mathfrak{m}^\rho \cap \mathfrak{a}$. Choose elements a_i $(i = 1, 2, \ldots, r)$ in $\mathfrak{a}$ such that $a_i \equiv \hat{a}_i (\mathfrak{m}^\rho \hat{A})$, and set $\mathfrak{a}' = (a_1, \ldots, a_r)A$. Then

$$a_i - \hat{a}_i \in \mathfrak{m}^\rho \hat{A} \cap \mathfrak{a}\hat{A} = (\mathfrak{m}^\rho \cap \mathfrak{a})\hat{A} \subseteq \mathfrak{m}\mathfrak{a}\hat{A},$$

whence $\hat{a}_i \in \mathfrak{a}'\hat{A} + \mathfrak{m}\mathfrak{a}\hat{A}$, $\mathfrak{a}\hat{A} \subseteq (\mathfrak{a}' + \mathfrak{m}\mathfrak{a})\hat{A}$. Hence $\mathfrak{a} \subseteq \mathfrak{a}' + \mathfrak{m}\mathfrak{a}$, and it follows that $\mathfrak{a} \subseteq \mathfrak{a}' + \mathfrak{m}^\nu \mathfrak{a} \subseteq \mathfrak{a}' + \mathfrak{m}^\nu$ for any positive integer ν. Finally, $\mathfrak{a} = \mathfrak{a}' = (a_1, \ldots, a_r)$.

THEOREM 2. (i) *Let $\mathfrak{z}$ be a prime ideal of A and $\hat{\mathfrak{z}}$ an isolated prime divisor of $\mathfrak{z}\hat{A}$, then* rank $\mathfrak{z} \geqq$ rank $\hat{\mathfrak{z}}$. (ii) *Let $\mathfrak{a}$ be an ideal of A, then* rank $\mathfrak{a} \geqq$ rank $\mathfrak{a}\hat{A}$. (iii) *In addition, if A is a local ring, then* dim $\mathfrak{a} =$ dim $\mathfrak{a}\hat{A}$.

Proof. (i) Put rank $\mathfrak{z} = r$, and take an ideal $\mathfrak{b}$ of A which is generated by r elements and has $\mathfrak{z}$ as one of its isolated prime divisors (**3**, p. 61, Theorem 8). Let $\mathfrak{b} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$ be an irredundant representation of $\mathfrak{b}$ as an intersection

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of primary ideals and let $\mathfrak{z}_i$ $(i = 1, 2, \ldots s)$ be the prime divisors of $\mathfrak{q}_i$; we assume $\mathfrak{z}_1 = \mathfrak{z}$. Then

$$\mathfrak{b}\hat{A} = \mathfrak{q}_1 \hat{A} \cap \ldots \cap \mathfrak{q}_s \hat{A}.$$

Let $\hat{\mathfrak{z}}_i$ be a prime divisor of $\mathfrak{q}_i \hat{A}$; then, as is well known, $\hat{\mathfrak{z}}_i \cap A = \mathfrak{z}_i$; and $\mathfrak{z}_i \hat{A}$ and $\mathfrak{q}_i \hat{A}$ have the same isolated prime divisors. From these observations it follows that $\hat{\mathfrak{z}}$ is an isolated prime divisor of $\mathfrak{b}\hat{A}$. Thus our claim is substantiated by Krull's Primidealkettensatz **(3**, p. 60, Theorem 7**)**.

(ii) Let $\mathfrak{z}$ be an isolated prime divisor of $\mathfrak{a}$ such that rank $\mathfrak{a}$ = rank $\mathfrak{z}$; and let $\hat{\mathfrak{z}}$ be an isolated prime divisor of $\mathfrak{z}\hat{A}$. Then it follows from the above observation that $\hat{\mathfrak{z}}$ is an isolated prime divisor of $\mathfrak{a}\hat{A}$, and

$$\text{rank } \mathfrak{z} \geqq \text{rank } \hat{\mathfrak{z}} \geqq \text{rank } \mathfrak{a}\hat{A}.$$

(iii) We have only to recall that $\hat{A}/\mathfrak{a}\hat{A}$ may be regarded as the completion of $A/\mathfrak{a}$; hence $\hat{A}/\mathfrak{a}\hat{A}$ and $A/\mathfrak{a}$ have the same dimension.

Next we shall consider the following two properties of a local ring A.

(α) For any prime ideal $\mathfrak{z}$ of A, rank $\mathfrak{z}$+dim $\mathfrak{z}$ = dim A.

(β) All the maximal chains of prime ideals of A have the same length.

Now (β) obviously implies (α), but the writer does not know whether conversely (α) implies (β). However, *in the case that A is complete, (α) and (β) are equivalent, and moreover A has the property (α) if and only if all the minimal prime ideals of A have the same dimension.* These facts are familiar and readily seen by the fundamental structure theorem for complete local rings due to I. S. Cohen.

COROLLARY. *If $\hat{A}$ has the property (α), so also has A.*

Proof. Let $\mathfrak{z}$ be a prime ideal of A, then

$$\dim A \geqq \text{rank } \mathfrak{z}+\dim \mathfrak{z} \geqq \text{rank } \mathfrak{z}\hat{A}+\dim \mathfrak{z}\hat{A} = \dim \hat{A} = \dim A.$$

This noteworthy fact was found by M. Nishi **(2)**.

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Hiroshima University

CERTAIN DIOPHANTINE EQUATIONS LINEAR IN ONE UNKNOWN

W. H. MILLS

1. Introduction. A. Brauer and R. Brauer **(2)** and Barnes **(1)** (following a method of Mordell **(6)**) have solved the Diophantine equation $x^2+y^2+c = xyz$ subject to the condition $(x, y)=1$. Independently, but using the same methods, I treated **(4)** the equation

$$x^2+y^2+ax+ay+1 = xyz,$$

and subsequently **(5)** gave a method of obtaining all integral solutions of

$$x^2\pm y^2+ax+by+c = xyz,$$

thereby generalizing **(2)**, **(1)**, and **(4)**. Recently Goldberg, Newman, Straus, and Swift **(3)** have treated the equation

$$ax^2+bxy+cy^2 = (p+qxy)z,$$

where a, b, c, p, and q are integers satisfying the divisibility conditions $a|(b, q)$ and $c|(b, q)$. In the present paper we combine the methods of **(3)** and **(5)**. This enables us to obtain the complete solution of the Diophantine equation

$$ax^2+bxy+cy^2+dx+ey+f = z(pxy+qx+ry+s), \tag{1}$$

where a, b, c, d, e, f, p, q, r, and s are integers satisfying the divisibility conditions $a|(b, d, p, q)$ and $c|(b, e, p, r)$, and $p \neq 0$. This generalizes all the previous results.

The solutions of (1) can be divided into classes, such that once one solution belonging to a given class is known the others can be determined recursively. In fact, after one solution is known, the others belonging to the same class can be determined explicitly by solving a pair of linear difference equations. Thus the problem is reduced to finding one solution in each class. All solutions in the same class have the same value of z. For sufficiently large numerical values of z there is a solution in each class for which either $x = -r/p$ or $y = -q/p$, or for which both sides of (1) vanish. This makes it easy to give necessary and sufficient conditions for (1) to have solutions for an infinite number of different values of z, and means that for large numerical values of z the solutions of (1) can be readily obtained. Since for particular values of z the equation (1) is a quadratic Diophantine equation in two variables, which can be solved in integers by classical methods, we see that the complete solution of (1) can always be obtained in a finite number of steps. It is not necessary to use classical quadratic equation methods to solve (1) for particular values of

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z. A method more in line with the spirit of this paper is developed in §4. For a fixed value of z there can be only a finite number of solution classes except in two simple special cases.

Goldberg, Newman, Straus, and Swift have pointed out **(3)** that if the divisibility conditions are not satisfied the solution has an entirely different character. The present methods give only an incomplete solution in this case.

2. Construction of α-sequences. Let us put

$$N(x, y) = ax^2+bxy+cy^2+dx+ey+f, \qquad D(x, y) = pxy+qx+ry+s,$$

where all the coefficients are rational integers, $p \neq 0$, and

$$a|(b, d, p, q), \qquad c|(b, e, p, r). \tag{2}$$

These conditions imply that $a \neq 0$ and $c \neq 0$. We consider the Diophantine equation

$$N(x, y) = zD(x, y). \tag{3}$$

Let $x = u_0$, $y = u_1$, $z = \alpha$ be an integral solution of (3). Then

$$N(x, u_1) = \alpha D(x, u_1) \tag{4}$$

is a quadratic equation in x with roots $x = u_0$ and $x = u_2$, where u_2 satisfies

$$a(u_0+u_2) = pu_1\alpha+q\alpha-bu_1-d. \tag{5}$$

It follows from (2) that u_0+u_2 is an integer. Therefore $x = u_2$, $y = u_1$, $z = \alpha$ is also an integral solution of (3). Continuing in this manner we obtain a sequence

$$\ldots, u_0, u_1, u_2, u_3, \ldots \tag{6}$$

where

$$a(u_{2n}+u_{2n+2}) = pu_{2n+1}\alpha+q\alpha-bu_{2n+1}-d, \tag{7}$$

and

$$c(u_{2n-1}+u_{2n+1}) = pu_{2n}\alpha+r\alpha-bu_{2n}-e. \tag{8}$$

It is clear that (6) can be extended infinitely far in either direction and that $x = u_{2n}$, $y = u_{2n\pm1}$, $z = \alpha$ are integral solutions of (3) for every integer n. We will call such a sequence an *α-sequence* of (3). We consider two α-sequences identical if they lead to the same solutions of (3).

We need the following result:

LEMMA. *If $|u_2| > |u_0|$, then either $pu_1+q = 0$ or*

$$|u_0| < |c/a|^{\frac{1}{2}}|u_1|+C,$$

where C is a constant depending only on the coefficients of $N(x, y)$ and $D(x, y)$.

Proof. Since u_0 and u_2 are the roots of (4), we have (5) and

$$au_0u_2 = cu_1^2+eu_1+f-\alpha(ru_1+s). \tag{9}$$

Eliminating α from (5) and (9) we obtain

$$au_2D(u_0, u_1)+au_0(ru_1+s) = cpu_1^2+Q(u_1),$$

where $Q(u_1)$ is a quadratic polynomial in u_1 with constant coefficients, i.e., coefficients that depend only on a, b, c, d, e, f, p, q, r, and s. Since $|u_2| \geqslant |u_0|$, we have

$$\begin{aligned}|cpu_1^2 + Q(u_1)| &\geqslant |au_0D(u_0, u_1)| - |au_0(ru_1 + s)| \\ &\geqslant |a(pu_1 + q)\, u_0^2| - 2|a(ru_1 + s)\, u_0|.\end{aligned}$$

Therefore, if $pu_1+q \neq 0$, the quadratic formula yields

$$\begin{aligned}|u_0| &\leqslant |pu_1 + q|^{-1}\{|(ru_1 + s)| + (|(ru_1 + s)|^2 + |a^{-1}(pu_1 + q)(cpu_1^2 + Q(u_1)|)^{\frac{1}{2}}\} \\ &\leqslant |c/a|^{\frac{1}{2}}|u_1| + C,\end{aligned}$$

where C is a positive constant depending only on the coefficients of $N(x, y)$ and $D(x, y)$. This proves the Lemma.

3. Classification of α-sequences. We will distinguish four types of α-sequences.

Type I. We will say that an α-sequence is a *type* I *sequence* if $D(u_{2k}, u_{2k+1}) = 0$ or $D(u_{2k}, u_{2k-1}) = 0$ for some integer k. It is clear that there exist type I sequences of (3) if and only if the system

$$D(x, y) = N(x, y) = 0 \tag{10}$$

has an integral solution. Each integral solution of (10) leads to an α-sequence of (3) for every integer α. Since $pac \neq 0$, (10) has at most four solutions, and so there are at most four type I α-sequences for any particular value of α.

Type II. We will say that an α-sequence is a *type* II *sequence* if

$$pu_{2k+1}+q = 0, \qquad ru_{2k+1}+s \neq 0 \tag{11}$$

for some integer k. We see that (11) implies $p|q$, $ps \neq qr$, and $N(u_{2k}, -q/p) = \alpha(ps - qr)/p$. Thus there exist type II sequences of (3) if and only if $p|q$, $ps \neq qr$, and the congruence

$$pN(x, -q/p) \equiv 0 \pmod{ps - qr}$$

has integral solutions. If type II sequences exist, then there exist α-sequences of type II for an infinite number of different values of α, namely for every integer α that can be represented in the form $pN(x, -q/p)/(ps - qr)$ with integral x.

Type IIA. We will say that an α-sequence is of *type* IIA if $pu_{2k} + r = 0$ and $qu_{2k} + s \neq 0$ for some integer k. Interchanging the roles of x and y interchanges the type II and the type IIA sequences. Therefore the discussion of type II sequences can be applied directly to the type IIA sequences.

It is clearly a straightforward matter to determine all type I, II, and IIA sequences for any given numerical values of the parameters a, b, c, d, e, f, p, q, r, and s.

Type III. If an α-sequence is not of type I, II, or IIA, we say that it is a *type* III *sequence.*

THEOREM 1. *There are α-sequences of type* III *for only a finite number of different values of α.*

Proof. Let $\{u_m\}$ be a type III sequence. Without loss of generality we suppose that u_1 is an element of $\{u_m\}$ of least absolute value. (This may involve interchanging the roles of x and y.) Furthermore we may suppose that $|u_2| \geq |u_0|$.

If $pu_1 + q = 0$ and $ru_1 + s = 0$, then $D(u_0, u_1) = 0$. Hence, since $\{u_m\}$ is neither of type I nor of type II, we have $pu_1 + q \neq 0$. Therefore $|u_0| \leq |c/a|^{\frac{1}{2}} |u_1| + C$ by the Lemma of §2. Also, since $|u_1|$ is minimal, we have $|u_1| \leq |u_0|$. We now distinguish two cases:

Case 1. $|pu_0 u_1| \leq 2|qu_0 + ru_1 + s|$. Here

$$|pu_0 u_1| \leq 2(|q|\,|c/a|^{\frac{1}{2}} + |r|)|u_1| + 2|q|C + 2|s|.$$

Hence if $u_1 \neq 0$ we have $|u_1| \leq |u_0| \leq D$ for some constant D, while if $u_1 = 0$ we have $|u_0| \leq C$. Thus both $|u_0|$ and $|u_1|$ are bounded, and hence there are only a finite number of possible values for $\alpha = N(u_0, u_1)/D(u_0, u_1)$.

Case 2. $|pu_0 u_1| > 2|qu_0 + ru_1 + s|$. Here $|D(u_0, u_1)| > \frac{1}{2}|pu_0 u_1|$, $u_0 \neq 0$, $u_1 \neq 0$, and so

$$|p\alpha| = \frac{|pN(u_0, u_1)|}{|D(u_0, u_1)|} < 2\frac{|au_0|}{|u_1|} + 2|b| + 2|c| + 2|d| + 2|e| + 2|f|,$$

which is bounded since $|u_0/u_1| \leq |c/a|^{\frac{1}{2}} + C$. This completes the proof of Theorem 1.

As a direct result of Theorem 1 and the elementary properties of type I, II, and IIA sequences we obtain:

THEOREM 2. *A necessary and sufficient condition for* (3) *to have integral solutions for an infinite number of different values of z is that either*

(i) *the system $N(x, y) = D(x, y) = 0$ has at least one integral solution, or*

(ii) *$p|q$, $ps \neq qr$, and the congruence $pN(x, -q/p) \equiv 0 \pmod{ps - qr}$ has at least one integral solution, or*

(iii) *$p|r$, $ps \neq qr$, and the congruence $pN(-r/p, y) \equiv 0 \pmod{ps - qr}$ has at least one integral solution.*

Special cases of Theorem 2 can be found in **(3)**, **(4)**, and **(5)**.

4. The α-sequences for a fixed value of α. We proved in the last section that, for any fixed values of the coefficients of $N(x, y)$ and $D(x, y)$, the possible

values of z are bounded except for a limited number of infinite classes of sequences, which can be readily determined. To obtain the complete solution of (3) we need an effective method of determining all α-sequences for a fixed value of α. In this section we will discuss such a method.

Let $\{u_n\}$ be an α-sequence of (3). Put $b' = b - p\alpha$, $d' = d - q\alpha$, $e' = e - r\alpha$, and $f' = f - s\alpha$. Then (3) becomes

$$ax^2 + b'xy + cy^2 + d'x + e'y + f' = 0,$$

and the divisibility conditions (2) yield

$$a|(b', d') \qquad c|(b', e'). \tag{12}$$

Furthermore (7) and (8) become

$$a(u_{2n} + u_{2n+2}) = -b'u_{2n+1} - d', \tag{13}$$

and

$$c(u_{2n-1} + u_{2n+1}) = -b'u_{2n} - e'. \tag{14}$$

If we eliminate u_1 from $a(u_0 + u_2) = -b'u_1 - d'$ and

$$au_0u_2 = cu_1^2 + e'u_1 + f',$$

which is obtained from (9), then we get $G(u_0, u_2) = 0$, where

$$G(u, v) = u^2 + Buv + v^2 + Du + Dv + F,$$

$B = 2 - b'^2/ac$, $D = (2cd' - b'e')/ac$, and

$$F = (cd'^2 + b'^2f' - b'd'e')/a^2c.$$

It follows from (12) that B and D are integers. Hence F must be an integer. From (13) and (14) we obtain

$$\begin{aligned} ac(u_{2n} + 2u_{2n+2} + u_{2n+4}) &= -b'c(u_{2n+1} + u_{2n+3}) - 2cd' \\ &= b'^2u_{2n+2} + b'e' - 2cd'. \end{aligned}$$

Therefore

$$u_{2n} + u_{2n+4} = -Bu_{2n+2} - D. \tag{15}$$

It follows that $x = u_0$ and $x = u_4$ are the roots of $G(x, u_2) = 0$, and hence

$$u_0u_4 = u_2^2 + Du_2 + F. \tag{16}$$

Now we may suppose, without loss of generality, that $|u_2| \leqslant |u_{2n}|$ for all n. In particular $|u_0| \geqslant |u_2|$ and $|u_4| \geqslant |u_2|$, and so we may write

$$u_0 = \epsilon u_2 + \delta, \qquad u_4 = \epsilon' u_2 + \delta',$$

where

$$\epsilon = \pm 1, \qquad \epsilon' = \pm 1, \qquad \epsilon u_2\,\delta \geqslant 0, \qquad \epsilon' u_2\,\delta' \geqslant 0. \tag{17}$$

Substituting in (15) and (16) we obtain

$$-Bu_2 - D = (\epsilon + \epsilon')u_2 + \delta + \delta', \tag{18}$$

and

$$u_2^2 + Du_2 + F = \epsilon\epsilon' u_2^2 + \epsilon\delta' u_2 + \epsilon'\delta u_2 + \delta\delta'. \tag{19}$$

Every non-zero term on the right hand side of (19) has the same sign by (17). Therefore

$$|Du_2 + F| \geqslant |\delta' u_2| + |\delta u_2|. \tag{20}$$

If $u_2 \neq 0$, then (20) yields $|\delta| + |\delta'| \leqslant |D| + |F|$, and there are only a finite number of possibilities for ϵ, ϵ', δ, and δ'. From (18) and (19) we see that for any fixed values of ϵ, ϵ', δ, and δ' there are at most two values of u_2 unless

$$\begin{aligned} &\epsilon + \epsilon' = -B, \quad \delta + \delta' = -D, \\ &\epsilon\epsilon' = 1, \quad \epsilon\delta' + \epsilon'\delta = D, \quad \delta\delta' = F. \end{aligned} \tag{21}$$

Thus, unless (21) has an integral solution, there are only a finite number of possible values for u_2. Each value of u_2 leads to at most one α-sequence. Hence if (21) has no integral solution, then there are at most a finite number of α-sequences, and they can all be found in a finite number of steps.

Suppose that (21) holds. Then $\epsilon = \epsilon' = \pm 1$.

(i) $\epsilon = \epsilon' = 1$. Here we have $B = -2$, $-D = \delta + \delta' = D$, and $\delta\delta' = F$. Hence $D = 0$. Ignoring the possibility $u_2 = 0$, which leads to at most one α-sequence, we obtain $\delta = \delta' = 0$ from $\delta + \delta' = 0$ and (17). Therefore

$$B = -2, \quad D = F = 0. \tag{22}$$

It is easily seen that (22) is equivalent to

$$4a(ax^2 + b'xy + cy^2 + d'x + e'y + f') = (2ax + b'y + d')^2. \tag{23}$$

Conversely suppose (23) holds. Then, since u_{2n} and u_{2n+2} are the roots of $(2ax + b'u_{2n+1} + d')^2 = 0$, we see that $u_{2n} = u_{2n+2}$. Similarly $u_{2n-1} = u_{2n+1}$. Therefore $u_m = u_{m+2}$ for all m, and so in this case every α-sequence is cyclic of period 1 or 2. Furthermore, since $B = -2$, we have $(b - p\alpha)^2 = 4ac$, and thus this type of behavior can occur for at most two values of α, namely $\alpha = (b \pm 2\sqrt{ac})/p$.

(ii) $\epsilon = \epsilon' = -1$. Here (21) yields $B = 2$, and so $b' = 0$. Conversely suppose $b' = 0$. Then $\alpha = b/p$. Here (13) gives us

$$u_{2n} + u_{2n+2} = -d'/a$$

for all n. Hence $u_{2n} = u_{2n+4}$. Similarly from (14) we obtain $u_{2n-1} = u_{2n+3}$. Therefore $u_m = u_{m+4}$ for all m. If $u_m = u_{m+2}$ for all m, then $u_0 = -d'/2a$, $u_1 = -e'/2c$, and so the conic

$$ax^2 + cy^2 + d'x + e'y + f' = 0$$

degenerates either to a single point or to a pair of intersecting straight lines. Therefore if $b' = 0$ every α-sequence is cyclic, and with at most one exception every α-sequence has period exactly 4. We have proved:

THEOREM 3. *If* (3) *has an infinite number of α-sequences for a fixed value of α, then either*

(i) $\alpha = (b \pm 2\sqrt{ac})/p$, $N(x, y) - \alpha D(x, y)$ *is a constant times a perfect square, and every α-sequence is of period* 1 *or* 2, *or*

(ii) $\alpha = b/p$, $N(x, y) - \alpha D(x, y)$ *has no xy term, and with at most one exception every α-sequence has period exactly* 4.

Using the methods of §3 and §4 all α-sequences of (3) can be found in a finite number of steps. In particular cases short cuts are frequently available, and sometimes it is but the work of a few lines to determine all α-sequences **(3; 4)**.

5. Cyclic α-sequences. We observed in the last section that

$$u_{2n} + Bu_{2n+2} + u_{2n+4} + D = 0. \tag{24}$$

Hence if $B = -2$, then u_{2n} is a quadratic function of n. If $B = 2$, then $b' = 0$, and we know that in this case $\{u_m\}$ is cyclic of period 1, 2, or 4. Now if $B \neq \pm 2$, then the general solution of the difference equation (24) is

$$u_{2n} = C_1\epsilon_1^n + C_2\epsilon_2^n - D/(B + 2),$$

where ϵ_1 and ϵ_2 are the roots of $x^2 + Bx + 1 = 0$, and C_1 and C_2 are arbitrary. Hence if $|B| > 2$, the sequence $\{u_{2n}\}$ is non-cyclic unless $C_1 = C_2 = 0$, in which case $u_{2n+2} = u_{2n}$. For $B = 1, 0$, and -1, ϵ_1 and ϵ_2 are the primitive 3rd, 4th, and 6th roots of unity respectively. Thus if $B = 1$, then $u_{2n+6} = u_{2n}$; if $B = 0$, then $u_{2n+8} = u_{2n}$; and if $B = -1$, then $u_{2n+12} = u_{2n}$. Since the identical statements hold for elements of the form u_{2n+1}, with D replaced by $(2ce' - b'd')/ac$ but with the same value of B, we have the following result:

THEOREM 4. *If an α-sequence is cyclic it has period* 1, 2, 3, 4, 6, 8, *or* 12.

All α-sequences have been determined in **(4)** for

$$x^2 + y^2 + dx + dy + 1 = xyz, \qquad 0 \leqslant d \leqslant 10.$$

They include cyclic sequences of periods 1, 2, 3, 4, and 6, as well as many non-cyclic sequences.

If $a = 1$, $b' = c = 2$, $d' = e' = 0$, $f = -10$, there is a cyclic sequence of period 8 with $u_0 = 2$, $u_1 = 1$:

$$\ldots, 2, 1, -4, 3, -2, -1, 4, -3, 2, 1, \ldots.$$

If $a = 1$, $b' = c = 3$, $d' = e' = 0$, $f = -13$, there is a cyclic sequence of period 12 with $u_0 = 2$, $u_1 = 1$:

$$\ldots, 2, 1, -5, 4, -7, 3, -2, -1, 5, -4, 7, -3, 2, 1, \ldots.$$

Thus we see that all the possible periods listed in Theorem 4 actually occur.

Additional examples and special cases of the results of this paper can be found in **(1; 2; 3; 4;** and **5)**.

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A CLASS OF ABELIAN GROUPS

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1. Introduction. If M is any finite set we define a *chain* on M as a mapping f of M into the set of ordinary integers. If $a \in M$ then $f(a)$ is the *coefficient* of a in the chain f. The set of all $a \in M$ such that $f(a) \neq 0$ is the *domain* $|f|$ of f. If $|f|$ is null, that is if $f(a) = 0$ for all a, then f is the *zero* chain on M. If M is null it is convenient to say that there is just one chain, a zero chain, on M.

The *sum* $f + g$ of two chains f and g on M is a chain on M defined by the following rule:

$$(f + g)(a) = f(a) + g(a), \qquad a \in M. \tag{1.1}$$

If M is null we take this to mean that the sum of the zero chain on M with itself is again the zero chain on M.

With this definition of addition the chains on M are the elements of an additive Abelian group $A(M)$. The zero element of $A(M)$ is the zero chain on M and the negative in $A(M)$ of a chain f on M is obtained from f by multiplying each coefficient $f(a)$ by -1. We define a *chain-group* on M as any subgroup of $A(M)$.

Let N be any chain-group on M. A chain f of N is an *elementary* chain of N (written f elc N) if it is non-zero and there is no non-zero $g \in N$ such that $|g|$ is a proper subset of $|f|$. If in addition the coefficients of f are restricted to the values 0, 1 and -1 we say that f is a *primitive* chain of N. We note that the negative of a primitive chain of N is another primitive chain of N.

We call N *regular* if for each elementary chain f of N there exists a primitive chain g of N such that $|g| = |f|$.

In this paper we study the properties of regular chain-groups. We find in particular that any finite graph has two associated regular chain-groups, and we relate the structure of these chain-groups to that of the graph. In discussing graphs we use the definitions and notation laid down in the introduction to **(4)**.

2. Cycles and coboundaries on a graph. Let G be any finite graph. If $S \subseteq E(G)$ we denote by $G \,.\, S$ that subgraph of G whose edges are the members of S and whose vertices are the ends in G of the members of S. We denote by $G : S$ that subgraph of G whose edges are the members of S and whose vertices are all the vertices of G. Clearly $G \,.\, S$ may be derived from $G : S$ by suppressing its *isolated* vertices, that is the vertices not ends of edges of $G : S$.

We denote by G ctr S the graph whose vertices are the *components* of $G : (E(G) - S)$ and whose edges are the members of S, the ends in G ctr S of an

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edge A being those components of $G : (E(G) - S)$ which contain as vertices the ends of A in G. We may regard G ctr S as obtained from G by contracting each component of $G : (E(G) - S)$ to a single point. We denote by $G \times S$ the graph obtained from G ctr S by suppressing its isolated vertices. These vertices are clearly those components of G whose edges all belong to $E(G) - S$.

If S is the set of edges of a circular path P in G we denote the graph $G \,.\, S$ by $G(P)$ and call it a *circuit* of G.

We call a graph a *bond* if it has just two vertices, no loops, and at least one link. Each link of course has the two vertices as its ends. A *bond of* G is a graph of the form $G \times S$ which is a bond.

Now let an orientation of G be given and let it be described by a function $\eta(A, a)$ as in **(4)**. We refer to chains on $V(G)$ and $E(G)$ as 0-*chains* and 1-*chains* on G respectively. We define their boundaries and coboundaries in the usual way. Thus the *boundary* ∂f of a 1-chain f is given by

$$(\partial f)(a) = \sum_{A \in E(G)} \eta(A, a)\, f(A), \tag{2.1}$$

and the *coboundary* δg of a 0-chain g by

$$(\delta g)(A) = \sum_{a \in V(G)} \eta(A, a)\, g(a). \tag{2.2}$$

If $E(G)$ is null we take (2.1) to mean that ∂f is the zero chain on $V(G)$. Similarly if $V(G)$ is null δg is the zero chain on $E(G)$. A *cycle* on G is a 1-chain whose boundary is the zero chain on $V(G)$.

The set of all cycles on the oriented graph G is clearly a chain-group $\Gamma(G)$ on $E(G)$. Another chain-group on $E(G)$ is the set $\Delta(G)$ of the coboundaries of the 0-chains on G. We proceed to show that $\Gamma(G)$ and $\Delta(G)$ are regular.

(2.3) *Let $G \,.\, S$ be any circuit of G. Then there is a primitive chain g of $\Gamma(G)$ such that $|g| = S$.*

Proof. There is a circular path $P = (a_0, A_1, \ldots, A_r, a_0)$ in G such that $G(P) = G \,.\, S$. Let g be a 1-chain of G defined as follows:

(i) If $A \notin S$ then $g(A) = 0$,

(ii) $g(A_i) = 1$ or -1 according as a_{i-1} is or is not the positive end of $A_i (0 < i \leqslant r)$.

Applying (2.1) we find that ∂g is a zero chain. Hence $g \in \Gamma(G)$.

If g is not an elementary chain of $\Gamma(G)$ there exists $k \in \Gamma(G)$ such that $|k|$ is a non-null proper subset of S. Then S has at least two elements. Hence by the definition of a circuit the elements of $|k|$ are links of G and some vertex of $G \,.\, |k|$ is an end of only one of them. This vertex must have a non-zero coefficient in ∂k, which is impossible. Accordingly g is elementary, and therefore primitive since its coefficients are restricted to the values 0, 1, and -1.

(2.4) *Suppose $S \subseteq E(G)$. Then S is the domain of an elementary chain of $\Gamma(G)$ if and only if $G \,.\, S$ is a circuit of G.*

Proof. Suppose g elc $\Gamma(G)$. We show that there is a circuit $G \cdot S$ of G such that $S \subseteq |g|$. If $G \cdot |g|$ has a loop this result is trivial. If not, each vertex of $G \cdot |g|$ is an end of at least two links of $G \cdot |g|$, by (2.1). Hence, starting at an arbitrary vertex a_0 of $G \cdot |g|$, we can construct a path

$$P = (a_0, A_1, a_1, A_2, a_2, \ldots)$$

of arbitrary length in $G \cdot |g|$ such that A_i and A_{i+1} are distinct for each i such that both exist as terms of P. We continue the path until some vertex b is repeated. Then the part of P extending from the first to the second occurrence of b is a circular path in $G \cdot |g|$ defining a circuit $G \cdot S$ such that $S \subseteq |g|$.

By (2.3) there exists $k \in \Gamma(G)$ such that $|k| = S \subseteq |g|$. Since g elc $\Gamma(G)$ it follows that $|g| = S$. Thus $G \cdot |g|$ is a circuit of G.

Since the converse result is contained in (2.3) the Theorem follows.

(2.5) *$\Gamma(G)$ is a regular chain-group.*

Proof. Suppose f elc $\Gamma(G)$. By (2.4) there is a circuit $G \cdot S$ of G such that $|f| = S$. Hence by (2.3) there is a primitive chain g of $\Gamma(G)$ such that $|g| = S = |f|$.

(2.6) *Let $G \times S$ be any bond of G. Then there is a primitive chain g of $\Delta(G)$ such that $|g| = S$.*

Proof. There are two distinct components X and Y of $G : (E(G) - S)$ such that in G each edge of S has one end in X and one in Y. Let f be the 0-chain on G such that $f(a) = 1$ if a is a vertex of X and $f(a) = 0$ otherwise. Write $g = \delta f$. Then $|g| = S$ by (2.2). Further the coefficients of G are restricted to the values 0, 1, and -1.

If g is not an elementary chain of $\Delta(G)$ there exists $k \in \Delta(G)$ such that $|k|$ is a non-null proper subset of S. Then X and Y are subgraphs of the same component, Z say, of $G : (E(G) - |k|)$. There is a 0-chain f on G such that $k = \delta f$. Since each edge of $|k|$ has both its ends in Z there are two vertices of Z having different coefficients in f. Since Z is connected it must have a link B whose ends have different coefficients in f. But then

$$k(B) = (\delta f)(B) \neq 0$$

by (2.2), which is impossible. Accordingly g is elementary, and therefore primitive since its coefficients are restricted to the values 0, 1, and -1.

(2.7) *Suppose $S \subseteq E(G)$. Then S is the domain of an elementary chain of $\Delta(G)$ if and only if $G \times S$ is a bond of G.*

Proof. Suppose g elc $\Delta(G)$. There is a 0-chain f on G such that $g = \delta f$. Since g is non-zero there is, by (2.2), a link A of G with ends a and b such that $f(a) \neq f(b)$. Write $f(a) = x$. Let W be the set of all $c \in V(G)$ such that $f(c) = x$. Let $G[U]$ be that component of $G[W]$ which has a as a vertex. (Here we use the notation of **(4)**). Let S be the set of all links of G having just one end in $G[U]$. Then $A \in S$. Moreover $S \subseteq |g|$, by (2.2).

Now $G[U]$ is one component of $G : (E(G) - S)$. Let Z be the component of $G : (E(G) - S)$ which has b as a vertex and let T be the set of all links of G having just one end in Z. Then $A \in T \subseteq S$. Let f' be that 0-chain on G in which the vertices of Z have coefficient 1 and all other vertices of G have coefficient 0. Then

$$A \in |\delta f'| = T \subseteq S \subseteq |g|,$$

by (2.2). Hence $|\delta f'| = |g|$, since g elc $\Delta(G)$, and therefore $T = S = |g|$.

We now see that each edge of S has one end in $G[U]$ and one in Z. Hence $G \times S$, that is $G \times |g|$, is a bond of G.

Since the converse result is contained in (2.6) the theorem follows.

(2.8) *$\Delta(G)$ is a regular chain-group.*

Proof. Suppose f elc $\Delta(G)$. By (2.7) there is a bond $G \times S$ of G such that $|f| = S$. Hence by (2.6) there is a primitive chain g of $\Delta(G)$ such that $|g| = S = |f|$.

3. Some operations on chain-groups. Let N be any chain-group on a set M. Let a subset S of M be chosen and let the coefficient of each member of S in each chain of N be multiplied by -1. The resulting chains are clearly the elements of a chain-group N' on M. We say that N' is obtained from N by *reorienting* the members of S.

Suppose M is the set of edges of an oriented graph G. By *reorienting* the members of S in G we mean interchanging positive and negative ends for each edge of G in S. By (2.1) and (2.2) the effect of this operation on the chain-groups $\Gamma(G)$ and $\Delta(G)$ is to reorient the members of S in each of them.

Properties of chain-groups which are invariant under reorientation are of special interest. Clearly one such property is that of regularity. We note also that the class of domains of elementary chain-groups is invariant under reorientation. In the case of $\Gamma(G)$ and $\Delta(G)$ the invariant properties correspond to properties of the underlying unoriented graph.

If $f \in N$ we define the *restriction* $f \,.\, S$ of f to S as that chain on S in which each $a \in S$ has the same coefficient as in f.

The restrictions to S of the chains of N are clearly the elements of a chain-group on S. We denote this chain-group by $N \,.\, S$. Another chain-group on S is the set of restrictions to S of those chains f of N for which $|f| \subseteq S$. We denote this by $N \times S$. If $T \subseteq S \subseteq M$ the following identities hold:

$$(3.1) \qquad (N \,.\, S) \,.\, T = N \,.\, T,$$

$$(3.2) \qquad (N \times S) \times T = N \times T,$$

$$(3.3) \qquad (N \,.\, S) \times T = (N \times (M - (S - T))) \,.\, T,$$

$$(3.4) \qquad (N \times S) \,.\, T = (N \,.\, (M - (S - T))) \times T.$$

Formulae (3.1) and (3.2) follow at once from the definitions. To prove (3.3) we observe that each side is the set of restrictions to T of those chains f of N

for which $|f| \cap (S - T)$ is null. We obtain (3.4) by writing $M - (S - T)$ for S in (3.3).

(3.5) *If N is regular then $N \,.\, S$ and $N \times S$ are regular.*

Proof. It is clear that the elementary and primitive chains of $N \times S$ are the restrictions to S of those elementary and primitive chains respectively of N whose domains are subsets of S. Hence $N \times S$ is regular.

Now suppose f elc $(N \,.\, S)$. There exists $g \in N$ such that $f = g \,.\, S$. Choose such a g so that $|g|$ has the least possible number of elements. Since N is regular it has a primitive chain h such that $|h| \subseteq |g|$. If $|h|$ does not meet S we can by adding h or $-h$ to g a sufficient number of times obtain $g' \in N$ such that $g' \,.\, S = f$ and $|g'|$ is a proper subset of $|g|$, contrary to the definition of g. We deduce that there is a non-zero chain $k = h \,.\, S$ of $N \,.\, S$ whose coefficients are restricted to the values 0, 1, and -1 and which satisfies $|k| \subseteq |h|$. Since f elc $(N \,.\, S)$ the chain k satisfies $|k| = |f|$ and is primitive. Thus $N \,.\, S$ satisfies the definition of a regular chain-group.

4. Dendroids and representative matrices. If f is a chain on a finite set M and n is an integer we denote by nf the chain obtained from f by multiplying each coefficient by n. It is clear that any chain-group containing f as an element contains also nf.

Let N be any chain-group on a finite set M.

We define a *dendroid* of N as a subset D of M such that D, but no proper subset of D, meets the domain of every non-zero chain of N. If the only element of N is the zero chain then the null subset of M is the only dendroid of N. In every other case M meets the domain of every non-zero chain of N and therefore some subset of M is a dendroid of N.

Suppose that D is a dendroid of N and that $a \in D$. There exists $f \in N$ such that $|f|$ is non-null and $|f| \cap (D - \{a\})$ is null. It follows that $|f| \cap D = \{a\}$ and hence that $f(a) \neq 0$. We can clearly choose f so that $f(a)$ is positive. We denote a choice of f for which $f(a)$ has the least possible positive value by $J^D{}_a$. There is only one such chain $J^D{}_a$, for the difference of two distinct ones would be a non-zero chain of N with a domain not meeting D.

(4.1) *$J^D{}_a$ is an elementary chain of N.*

Proof. Suppose k is a non-zero chain of N such that $|k|$ is a proper subset of $|J^D{}_a|$. Write $J^D(a) = m$ and $k(a) = n$. Since $D \cap |k|$ is non-null we have $n \neq 0$. The chain $nJ^D{}_a - mk$ of N is zero since its domain does not meet D. Hence $|k| = |J^D{}_a|$, contrary to the definition of k.

(4.2) *If N is regular $J^D{}_a$ is primitive*

Proof. By (4.1) and the regularity of N there is a primitive chain g of N such that $|g| = |J^D{}_a|$. Replacing g by its negative if necessary we can arrange that $g(a) = 1$. Then by the definition and uniqueness of $J^D{}_a$ we have $g = J^D{}_a$.

(4.3) *Suppose N is regular and has a non-null dendroid D. Then for each chain J of N we have*

$$J = \sum_{a \in D} J(a)\, J^D{}_a.$$

Proof. Write

$$J' = J - \sum_{a \in D} J(a)\, J^D{}_a.$$

It is clear, by (4.2), that $|J'|$ does not meet D. Hence J' is a zero chain.

In the rest of this section we suppose that the set M is non-null. We enumerate its elements as $a_1, \ldots, a_n$. If f is any chain on M we refer to the row-vector $\{f(a_1), \ldots, f(a_n)\}$ as the *representative vector* of f with respect to the chosen enumeration. Suppose R is a matrix of r rows and n columns whose elements are integers and whose rows are linearly independent. Then the set of chains on M whose representative vectors are the linear combinations of the rows of R with integral coefficients are the elements of a chain-group on M. If this chain-group is N we say that R is a *representative matrix* of N with respect to the chosen enumeration of the elements of M.

By the general theory of Abelian groups every chain-group on M having at least one non-zero element has a representative matrix. If N is a regular chain-group of this kind we may form a representative matrix R as follows. We select a dendroid D, necessarily non-null, and take as the rows of R the representative vectors of the corresponding chains $J^D{}_a$. It is easily seen that these vectors are linearly independent. It then follows from (4.3) that R is a representative matrix of N. We say that the representative matrix R thus constructed is *associated* with the dendroid D.

Suppose we have a representative matrix R of N, where N is not necessarily regular. Then if $S \subseteq M$ we denote by $R(S)$ the submatrix of R constituted by those columns of R which correspond to members of S. If $R(S)$ is square we denote its determinant by $\det R(S)$.

(4.4) *Let R be an r-rowed representative matrix of N. Then a subset S of M is a dendroid of N if and only if it has just r elements and is such that* $\det R(S) \neq 0$.

Proof. If the rank of $R(S)$ is less than r some linear combination of the rows of R with integral coefficients not all zero has only zeros in the columns corresponding to members of S. The corresponding chain of N is non-zero and has a domain not meeting S. Hence S is not a dendroid of N. In particular no dendroid of N has fewer than r elements.

If the rank $R(S)$ is r there is a subset T of S of just r elements such that $\det R(T) \neq 0$. Then the rows of $R(T)$ are linearly independent. Consequently T meets the domain of each non-zero chain of N and so some subset of T is a dendroid of N. This subset must be T itself since a dendroid of N has at least r elements. We conclude that S is a dendroid of N if it has r elements but not if it has more than r.

It follows from (4.4) that all the dendroids of N have the same number r of elements, and that each representative matrix of N has r rows. We call r the *rank* of N and denote it also by $r(N)$. The Theorem does not of course apply to the case in which N consists solely of the zero chain. In that case we write $r(N) = 0$. Then the only dendroid of N has $r(N)$ elements.

(4.5) *Let R be a matrix of r rows and n columns whose elements are integers and whose rows are linearly independent. Let M be a finite set of n elements. Then R is a representative matrix of a regular chain-group on M if and only if the determinants of its square submatrices of order r are restricted to the values* 0, 1, *and* -1.

Proof. Suppose first that R is a representative matrix of a regular chain-group N on M. Let $R(S)$ be any square submatrix of R of order r.

If S is not a dendroid of N then $\det R(S) = 0$, by (4.4). If S is a dendroid of N let R' be a representative matrix of N associated with S, and corresponding to the same enumeration of M as R. The rows of R' must be linear combinations of the rows of R with integral coefficients. Hence there is a square matrix P of order r whose elements are integers and which satisfies $R' = PR$. This implies $R'(S) = P \times R(S)$ and hence

$$\det R'(S) = \det P \,.\, \det R(S).$$

Now $\det R'(S) = \pm 1$, by the definition of R'. Since P and $R(S)$ are matrices of integers it follows that $\det R(S) = \pm 1$.

Conversely, suppose that the square submatrices of R of order r have determinants restricted to the values 0, 1, and -1. We fix an enumeration of the elements of M. There is a chain-group N on M whose representative matrix with respect to this enumeration is R.

Let f be any elementary chain of N. Let a be any member of $|f|$ and E any dendroid of $N\,.\,(M - |f|)$. Then if a chain h of N has a domain not meeting $E \cup \{a\}$ its domain must be a subset of $|f| - \{a\}$. Since f is elementary this is possible only if h is zero. We conclude that some subset D of $E \cup \{a\}$ is a dendroid of N. Since D must meet $|f|$ we have $D \cap |f| = \{a\}$.

By (4.4) and the restriction imposed on R we have $\det R(D) = \pm 1$. Hence the reciprocal of $R(D)$ is a matrix of integers. We write $R' = (R(D))^{-1}R$. The rows of R' are linear combinations, with integral coefficients, of the rows of R and are therefore representative vectors of chains of N. But $R'(D)$ is a unit matrix. Hence there is a chain g of N such that $g(a) = 1$ and $|g| \cap D = \{a\}$. Then $f - f(a)g$ is a zero chain since its domain does not meet D. Accordingly $f = f(a)g$.

Keeping $|f|$ fixed we may select f so that the highest common factor of its non-zero coefficients is as small as possible. With this choice of f the result just obtained requires $f(a) = \pm 1$. Since this is true for each $a \in |f|$ the chain f is then primitive. Thus N satisfies the definition of a regular chain-group.

(4.6) *Let R be a matrix of r rows and $n > r$ columns, whose elements are integers and in which the square submatrix A constituted by some r columns is unit matrix. Let the submatrix of R constituted by the remaining $n - r$ columns be B. Let M be any set of n elements. Then R is a representative matrix of a regular chain-group on M if and only if the determinants of the square submatrices of B are restricted to the values* $0, 1,$ *and* -1.

Proof. There is a 1–1 correspondence, q say, between the square submatrices of B and those square submatrices other than A of R which are of order r. If C is a square submatrix of B the corresponding submatrix qC of R is made up of those columns of B which contain elements of C and those columns of A which have only zeros in the rows of R meeting C. It is clear from this definition that $\det qC = \pm \det C$. Since the rows of A, and therefore the rows of R, are linearly independent the Theorem now follows from (4.5).

If R is a representative matrix of a regular chain-group N and R' is the transpose of R then the number $C(N)$ of dendroids of N is given by the formula

$$C(N) = \det (RR'). \tag{4.7}$$

This follows from (4.4) and (4.5), with the help of the well-known formula for the determinant of the product of two matrices of types (r, n) and (n, r).

5. Dual regular chain-groups. Two chains f and g on a finite set M are *orthogonal* if

$$\sum_{a \in M} f(a)\, g(a) = 0.$$

If M is null we take this to mean that the zero chain on M is self-orthogonal.

If N is a chain-group on M then these chains on M which are orthogonal to all the chains of N evidently constitute a chain-group on N. We denote this chain-group by N^* and call it the *dual* of N.

The *zero* chain-group on M includes only the zero chain. The *complete* chain-group on M includes all the chains on M. It is clear that these chain-groups are regular and that each is the dual of the other.

If N is a regular chain-group on M which is neither zero nor complete we may construct N^* as follows. We choose arbitrarily a dendroid D of N and denote by R a representative matrix of N associated with D. If $r(N) = r$ we may adjust the notation so that $R(D)$ is a unit matrix occupying the first r columns of R. We denote by B the matrix constituted by the remaining columns of R, which we suppose s in number. Now let T be the matrix of s rows and $r + s$ columns such that the submatrix formed by the first r columns is the negative of the transpose of B and the remaining s columns constitute a unit matrix. Let N_1 be the chain-group on M which has T as a representative matrix with respect to the chosen enumeration of M. By (4.6) N_1 is regular.

If $b \in M - D$ we denote by K_b that chain of N_1 which has a row of T as a representative vector and which satisfies $K_b(b) = 1$. It is clear that K_b is orthogonal to $J^D{}_a$ for each $a \in D$ and each $b \in M - D$. Hence by (4.3) the chains K_b are orthogonal to all the chains of N and therefore belong to N^*. It follows that $N_1 \subseteq N^*$. Now suppose N^* has a chain J not belonging to N_1. Write

$$J' = J - \sum_{b \in (M-D)} J(b)\, K_b.$$

The chain J' of N is orthogonal to each of the chains $J^D{}_a$ and its domain is a subset of D. It is therefore a zero chain. It follows that J belongs to N_1, contrary to supposition. We have thus proved that $N_1 = N^*$.

A similar argument in which the roles of the $J^D{}_a$ and the K_b are interchanged shows that R is a representative matrix of $(N^*)^*$ and hence that $(N^*)^* = N$. We now have

(5.1) *If N is a regular chain-group then N^* is regular and $(N^*)^* = N$.*

(5.2) *If N is a regular chain-group on a set M then the dendroids of N^* are the complements in M of the dendroids of N.*

Proof. Let D be any dendroid of N. If N is zero or complete it is clear that $M - D$ is a dendroid of N^*. Otherwise we form the matrix T as in the above construction. Since T is a representative matrix of N^* and $\det T(M - D) = 1$ it follows from (4.4) that $M - D$ is a dendroid of N^*. Replacing N by N^* in this result, and using (5.1), we find also that if $M - D$ is a dendroid of N^* then D is a dendroid of N.

Suppose N is a regular chain-group on a set M and that S is a subset of M. Then a chain g on S is orthogonal to every chain of $N \,.\, S$ if and only if it is of the form $f \,.\, S$, where $f \in N^*$ and $|f| \subseteq S$. We thus have

$$(N \,.\, S)^* = N^* \times S. \tag{5.3}$$

By writing N^* for N in (5.3) and using (5.1) we obtain also

$$(N \times S)^* = N^* \,.\, S. \tag{5.4}$$

(5.5) *Let G be a finite graph and let $\Gamma(G)$ and $\Delta(G)$ be defined in terms of the same orientation of G. Then $(\Delta(G))^* = \Gamma(G)$.*

Proof. If G has no edge the result is trivial. In the remaining case a 1-chain g on G is orthogonal to all the chains of $\Delta(G)$ if and only if

$$\sum_{A \in E(G)} \left\{ g(A) \sum_{a \in V(G)} \eta(A, a)\, f(a) \right\} = 0$$

for arbitrary integers $f(a)$. This is so if and only if

$$\sum_{A \in E(G)} \eta(A, a)\, g(A) = 0$$

for each $a \in V(G)$, that is, if and only if $g \in \Gamma(G)$.

The dendroids of a chain-group depend only on the domains of the chains of the group and are therefore invariant under reorientation. Hence if G is a finite graph and $\Delta(G)$ is its group of coboundaries with respect to some fixed orientation we may expect the dendroids of $\Delta(G)$ to be interpretable in terms of the structure of G only.

If H and K are two subgraphs of G we define their *intersection* $H \cap K$ as that subgraph of G whose edges and vertices are the common edges and vertices respectively of H and K. A *forest* is a graph which has no circuit. A *tree* is a connected forest. A *spanning forest* of G is a subgraph of G of the form $G : S$ whose intersection with each component of G is a tree.

(5.6) *Let G be a finite graph with a given orientation and let S be a subset of $E(G)$. Then S is a dendroid of $\Delta(G)$ if and only if $G : S$ is a spanning forest of G.*

Proof. Suppose $G : S$ is not a spanning forest of G. If $G : S$ has a circuit then $E(G) - S$ is not a dendroid of $\Gamma(G)$, by (2.4), and therefore S is not a dendroid of $\Delta(G)$, by (5.2) and (5.5). If $G : S$ has no circuit its intersection with each component of G is a forest. Hence there must be a component H of G such that $H \cap (G : S)$ is not connected. Let K be any component of $H \cap (G : S)$. Let f be the 0-chain on G such that $f(a) = 1$ if a is a vertex of K and $f(a) = 0$ otherwise. Then the chain δf is non-zero and its domain does not meet S. Again we find that S is not a dendroid of $\Delta(G)$.

Conversely suppose S is not a dendroid of $\Delta(G)$. Assume that $G : S$ is a spanning forest of G. Let f be any 0-chain on G such that δf is non-zero. Then some component H of G has two vertices a and b such that $f(a) \neq f(b)$. Since $H \cap (G : S)$ is a tree there are two vertices c and d of $H \cap (G : S)$, joined by an edge of S, such that $f(c) \neq f(d)$. Hence S meets $|\delta f|$. We deduce that some proper subset T of S is a dendroid of $\Delta(G)$. Choose $e \in S - T$ and write $Q = E(G) - T$. Now Q is a dendroid of $\Gamma(G)$, by (5.2). The non-zero element $J^Q{}_e$ of $\Gamma(G)$ satisfies $|J^Q{}_e| \subseteq S$. Hence, by (2.4), $G : S$ has a circuit, contrary to our assumption. We deduce that in fact $G : S$ is not a spanning forest of G. The Theorem follows.

6. Conformity. Let f and g be chains on a finite set M. We say that f *conforms* to g if the following condition is satisfied: if $f(a) \neq 0$ then $g(a)$ is non-zero and has the same sign as $f(a)$. Conformity is clearly a transitive relation.

(6.1) *If N is a regular chain-group and f is a non-zero chain of N then there exists a primitive chain of N conforming to f.*

Proof. If possible choose f so that the Theorem fails and $|f|$ has the least number of elements consistent with this condition. Since N is regular it has a primitive chain h such that $|h| \subseteq |f|$. Choose $a \in |h|$ so that $f(a)$ has the least possible absolute value. Replacing h by its negative if necessary, we arrange that $h(a) = 1$. Write $k = f - f(a)h$. Clearly k conforms to f. If k is a zero

chain then either h or $-h$ conforms to f. If k is non-zero there is a primitive chain g of N conforming to k, and therefore to f, since $|k|$ is a proper subset of $|f|$. In each case the definition of f is contradicted.

(6.2) *If N is a regular chain-group then each non-zero chain of N can be represented as a sum of primitive chains of N each conforming to it.*

Proof. If $f \in N$ let $Z(f)$ be the sum of the absolute values of the coefficients of f. If possible choose a non-zero $f \in N$ for which the Theorem fails and $Z(f)$ has the least value consistent with this condition. By (6.1) there is a primitive chain g of N conforming to f. Clearly $f - g$ conforms to f and $Z(f - g) < Z(f)$. By the latter result $f - g$ is either a zero chain or a sum of primitive chains of N conforming to it. But chains conforming to $f - g$ conform also to f. Hence the Theorem is true for f and we have a contradiction.

Let f and g be chains on a finite set M and let q be an integer >1. We say that g is a *q-representative* of f if the following conditions are satisfied:

(i) $g(a) \equiv f(a) \pmod{q}$ for each $a \in M$,

(ii) $|g(a)| < q$ for each $a \in M$.

(6.3) *If N is a regular chain-group on a set M and $f \in N$ then for each integer $q > 1$ some q-representative of f is a chain of N.*

Proof. Let f be any chain of N and q any integer >1. There is at least one $g \in N$ satisfying (i). For any such g we denote by $Y(g)$ the number of elements a of M for which $|g(a)| \geqslant q$. We choose a particular g satisfying (i) so that $Y(g)$ has the least possible value.

If $Y(g) > 0$ choose $b \in M$ such that $|g(b)| \geqslant q$. By (6.2) there is a primitive chain h of N conforming to g and such that $h(b) = \pm 1$. Write $g' = g - qh$. Clearly g' satisfies (i). Moreover we have

(1) $$|g'(b)| < |g(b)|,$$

(2) if $$|g(a)| < q \text{ then } |g'(a)| < q.$$

If $|g'(b)| \geqslant q$ we repeat the process with g' replacing g and with the same choice of b. Proceeding in this way we eventually obtain a chain g_1 of N which satisfies (i) and is such that $Y(g_1) < Y(g)$. This contradicts the definition of g. We conclude that $Y(g) = 0$, that is, g is a q-representative of f.

This Theorem is proved for the cycle-group of an oriented graph in **(3)**. For applications of it to the theory of graphs see **(3)** and **(4**, pp. 83–84**)**.

7. Homomorphisms. Let N be a regular chain-group on a set M. A *homomorphism* of N (into I) is a mapping ϕ of N into the set I of integers such that

(7.1) $$\phi(f + g) = \phi(f) + \phi(g)$$

for arbitrary chains f and g of N. This implies that $\phi(f) = 0$ if f is the zero chain. Hence $\phi(-f) = -\phi(f)$ for each $f \in N$.

For arbitrary chains f and g on M we write

$$(f \cdot g) = \sum_{a \in M} f(a)\, g(a). \tag{7.2}$$

If M is null we take this to mean $(f \cdot g) = 0$.

A *solution* of ϕ is a chain g on M such that $(f \cdot g) = \phi(f)$ for each $f \in N$. In this section we study the solutions of the homomorphisms of N. We need the following definitions.

If $f \in N$ we define $P(f)$ as the set of all $a \in M$ such that $f(a) > 0$. We then write

$$\beta(f) = \sum_{a \in P(f)} f(a). \tag{7.3}$$

If $P(f)$ is null we take $\beta(f)$ to be 0. We call f a *positive* chain of N if $P(f) = |f|$.

(7.4) *Let ϕ be any homomorphism of N and a any element of M. Then either $\{a\}$ is the domain of a chain of N or there is a homomorphism ϕ_a of $N \cdot (M - \{a\})$ such that*

$$\phi_a(f \cdot (M - \{a\})) = \phi(f)$$

for each $f \in N$.

Proof. Suppose $\{a\}$ is not the domain of a chain of N. Then no two distinct chains of N have the same restriction to $M - \{a\}$, for otherwise the domain of their difference would be $\{a\}$. Hence there is a unique mapping ϕ_a of $N \cdot (M - \{a\})$ into I such that

$$\phi_a(f \cdot (M - \{a\})) = \phi(f)$$

for each $f \in N$. It is easily verified that ϕ_a is a homomorphism.

(7.5) *If ϕ is any homomorphism of N and f is a chain of N such that $\phi(f) > \beta(f)$ then there is a primitive chain g of N conforming to f such that $\phi(g) > \beta(g)$.*

Proof. The chain f is necessarily non-zero. Hence by (6.2) it is a sum $f_1 + f_2 + \ldots + f_s$ of primitive chains f_i of N conforming to f. If the Theorem is false, $\phi(f_i) \leqslant \beta(f_i)$ for each of these. Then by addition we have $\phi(f) \leqslant \beta(f)$, contrary to hypothesis.

(7.6) *If ϕ is any homomorphism of N, a an element of M, and f a chain of N such that $f(a) \neq 0$ and*

$$\phi(f) - \beta(f) + \epsilon f(a) > 0,$$

where ϵ is 1 or -1, then there is a primitive chain g of N conforming to f such that either $\phi(g) > \beta(g)$ or g satisfies the equations $\phi(g) = \beta(g)$ and $g(a) = \epsilon$.

Proof. By (6.2) f is a sum $f_1 + f_2 + \ldots + f_s$ of primitive chains f_i of N each conforming to f. There must be just $|f(a)|$ of these such that $|f_i(a)| = 1$.

If one of the f_i satisfies $\phi(f_i) > \beta(f_i)$ the Theorem is true. In the remaining case we have by addition $\phi(f) - \beta(f) \leqslant 0$. But

$$\phi(f) - \beta(f) + \epsilon f(a) > 0.$$

One consequence of this is that $f(a)$ has the same sign as ϵ, whence it follows that the $|f(a)|$ chains f_i satisfying $|f_i(a)| = 1$ satisfy also $f_i(a) = \epsilon$. Another consequence is that at most $|f(a)| - 1$ of the chains f_i satisfy

$$\phi(f_i) - \beta(f_i) < 0.$$

Combining these results we see that one of the chains f_i satisfies both $\phi(f_i) = \beta(f_i)$ and $f_i(a) = \epsilon$.

(7.7) *Let ϕ be any homomorphism of N. Then in order that ϕ shall have a solution whose coefficients are restricted to the values* 0 *and* 1 *it is necessary and sufficient that $\phi(g) \leqslant \beta(g)$ for each primitive chain g of N.*

Proof. Let us call a solution of a homomorphism *limited* if its coefficients are restricted to 0 and 1.

The theorem is trivially true if M is null. Assume as an inductive hypothesis that it is true whenever the number $\alpha(M)$ of elements of M is less than some positive integer q. Consider the case $\alpha(M) = q$.

Suppose there is a primitive chain g of N such that $\phi(g) > \beta(g)$. Then any chain h on M with coefficients restricted to the values 0 and 1 satisfies

$$(g \,.\, h) \leqslant \beta(g) < \phi(g).$$

Hence no limited solution of ϕ exists.

Conversely suppose ϕ has no limited solution. Assume there is no primitive chain g of N such that $\phi(g) > \beta(g)$. It may happen that each $a \in M$ constitutes the domain of a chain of N. Then, since N is regular, there is for each $a \in M$ a chain f_a of N such that $f_a(a) = 1$ and $f_a(b) = 0$ if $b \neq a$. We define a chain h on M, with coefficients restricted to the values 0 and 1, by writing $h(a) = \phi(f_a)$ for each $a \in M$. Then for each $f \in N$ we have

$$(f \,.\, h) = ((\sum_{a \in M} f(a) f_a) \,.\, h) = \sum_{a \in M} f(a)(f_a \,.\, h)$$
$$= \sum_{a \in M} f(a)\, \phi(f_a) = \phi(f).$$

Thus h is a limited solution of ϕ. But this is impossible.

We deduce that there exists $a \in M$ such that $\{a\}$ is not the domain of a chain of N. We define ϕ_a as in (7.4). There is no limited solution of ϕ_a, for such a solution would be the restriction to $M - \{a\}$ of a limited solution d of ϕ satisfying $d(a) = 0$. Hence, by the inductive hypothesis and (3.5) there exists $f \in N$ such that

$$\phi_a(f \,.\, (M - \{a\})) - \beta(f \,.\, (M - \{a\})) > 0.$$

If $f(a) \leqslant 0$ it follows that $\phi(f) - \beta(f) > 0$. By (7.5) this is contrary to our assumptions. Hence $f(a) > 0$ and we have

$$\phi(f) - \beta(f) + f(a) > 0.$$

By (7.6) and our assumptions it follows that there is a chain j of N such that

(i) $$\phi(j) - \beta(j) = 0 \text{ and } j(a) = 1.$$

Now let ψ be the homomorphism of N defined by $\psi(f) = \phi(f) - f(a)$ for each $f \in N$. The homomorphism ψ_a of $N.(M - \{a\})$ has no limited solution, for such a solution would be a restriction to $M - \{a\}$ of a limited solution d of ϕ such that $d(a) = 1$. Hence by the inductive hypothesis and (3.5) there exists $f \in N$ such that

$$\phi(f) - f(a) - \beta(f.(M - \{a\})) > 0.$$

If $f(a) \geqslant 0$ this gives $\phi(f) - \beta(f) > 0$. By (7.5) this is contrary to our assumptions. Hence $f(a) < 0$ and

$$\phi(f) - \beta(f) - f(a) > 0.$$

By (7.6) and our assumptions it follows that there exists $k \in N$ such that

(ii) $$\phi(k) - \beta(k) = 0 \text{ and } k(a) = -1.$$

It follows from (i) and (ii) that $\phi(j + k) - \beta(j + k) > 0$. This is contrary to our assumptions, by (7.5). This completes the proof for the case $\alpha(M) = q$. The general theorem follows by induction.

(7.8) *Let ϕ be any homomorphism of N. Then ϕ has a solution whose coefficients are all non-negative if and only if $\phi(f) \geqslant 0$ for each positive primitive chain f of N.*

Proof. N has only a finite number of primitive chains. Hence we can find an integer $q > 0$ such that $\phi(f) < q$ for each primitive chain f of N.

Choose a set U, the union of $\alpha(M)$ disjoint sets U_a, one for each $a \in M$. Each U_a is to have just q elements. If $k \in N$ we denote by k' the chain on U in which the coefficient of each element of U_a is $k(a)$, for each $a \in M$. The chains k' constitute a chain-group N' on U. Elementary and primitive chains of N' correspond respectively to elementary and primitive chains of N. Hence N' is regular. There is a homomorphism ϕ' of N' such that $\phi'(k') = \phi(k)$ for each $k \in N$.

If $\phi(f) < 0$ for some positive primitive chain f of N it is clear that ϕ has no solution whose coefficients are all non-negative.

In the remaining case we have $\phi'(g') \leqslant \beta(g')$ for each primitive chain g' of N'. This follows from the definition of N' if $\beta(g') > 0$. In the remaining case $-g'$ corresponds to a positive chain $-g$ of N, and so $\phi'(g') = -\phi(-g) \leqslant 0 =$

$\beta(g')$. Hence by (7.7) ϕ' has a limited solution h'. There is a corresponding solution h of ϕ defined by

$$h(a) = \sum_{c \in U_a} h'(c), \quad a \in M.$$

The coefficients in h are all non-negative.

(7.9) *If $a \in M$ then either N or N^* has a positive primitive chain f such that $a \in |f|$.*

Proof. By (6.2) it is sufficient to show that either N or N^* has a positive chain f such that $a \in |f|$.

Let ϕ be the homomorphism of N such that $\phi(f) = -f(a)$ for each $f \in N$. If $\{a\}$ is the domain of a chain of N the Theorem is clearly true. If not we define ϕ_a as in (7.4). Then, if ϕ_a has a solution h' with coefficients all non-negative, let h be the chain on M such that $h(a) = 1$ and $h \,.\, (M - \{a\}) = h'$. Then $(f \,.\, h) = 0$ for each $f \in N$ and so h is a positive chain of N^*. If no such solution h' exists, then by (7.8) there exists $f \in N$ such that $f \,.\, (M - \{a\})$ is positive and $-f(a) = \phi(a) < 0$. Then f is a positive chain of N such that $a \in |f|$. In either case the Theorem is true.

8. Some applications to graph theory. Let G be a graph taken with a fixed orientation.

A *directed bond* of G is a bond $G \times S$ of G such that the positive ends of the edges of S all belong to the same component of $G : (E(G) - S)$. A *directed circuit* of G is a circuit $G \,.\, S$ of G defined by a circular path in which each edge is immediately succeeded by its positive end. Using (2.4) and (2.7) we may verify that the subsets S of $E(G)$ such that $G \times S$ is a directed bond or $G \,.\, S$ a directed circuit of G, are the domains of the positive primitive chains of $\Delta(G)$ and $\Gamma(G)$ respectively. If we apply this to (5.5) and (7.9) we obtain the following graph-theoretical result.

(8.1) *Any edge of G is an edge of some directed bond or of some directed circuit of G.*

In conclusion we show how (7.8) may be applied to obtain a known theorem concerning the 1-factors of even graphs **(1; 2)**.

We suppose henceforth that G is *even*, that is, the set $V(G)$ falls into two disjoint subsets V_1 and V_2 such that each edge of G has one end in V_1 and the other in V_2. We fix an orientation by taking the positive end of each edge in V_2. If $a \in V(G)$ we write $\sigma(a) = 1$ or -1 according as a is in V_2 or V_1. We call G *balanced* if each component has the same number of vertices in V_1 as in V_2. The decomposition $\{V_1, V_2\}$ of $V(G)$ is unique within each component of G, apart from the order of V_1 and V_2. Hence if G is balanced for one such decomposition it is balanced for all of them.

A 1-*factor* of G is a subgraph $G : F$ of G such that each vertex of G is an end of just one edge of F. It is clear that a graph which is not balanced has no 1-factor. For balanced graphs we prove the following theorem.

(8.2) *Suppose G balanced. Then G has a 1-factor if and only if there is no subset U of V_1 such that the set of all vertices of V_2 joined by edges of G to vertices of U has fewer members than U.*

Proof. If such a subset U of V_1 exists it is clear that G has no 1-factor.

Conversely suppose G has no 1-factor. Then for each $g \in \Delta(G)$ we write

$$\phi(g) = \sum_{a \in V(G)} \sigma(a) f(a), \tag{i}$$

where f is any 0-chain on G such that $\delta f = g$. If f_1 and f_2 are two such 0-chains and $\phi_1(g)$ and $\phi_2(g)$ are the corresponding values of $\phi(G)$ we have

$$\phi_1(g) - \phi_2(g) = \sum_{a \in V(G)} \sigma(a)(f_1(a) - f_2(a)). \tag{ii}$$

Now $\delta(f_1 - f_2) = \delta(f_1) - \delta(f_2)$, which is the zero 1-chain on G. Hence by (2.2) $f_1(a) - f_2(a)$ is the same for all vertices a of any one component of G. Since G is balanced it follows from (ii) that $\phi_1(g) = \phi_2(g)$. Hence $\phi(g)$ is uniquely defined for each $g \in \Delta(G)$. It is now clear that ϕ is a homomorphism of $\Delta(G)$.

Suppose ϕ has a solution h whose coefficients are all non-negative. By considering the coboundaries $\delta(f)$ such that f has only one non-zero coefficient we find that

$$\sum_{A \in E(G)} \eta(A, a)\, h(A) = \sigma(a) \tag{iii}$$

for each $a \in V(G)$. But $\eta(A, a)\, \sigma(a) \geqslant 0$ for each a, A. It follows that $h(A)$ is 0 or 1 for each A and that the edges for which $h(A) = 1$ define a 1-factor of G. This contradicts our supposition. Hence by (7.8) there is a positive primitive chain k of $\Delta(G)$ such that $\phi(k) < 0$.

Now $G \times |k|$ is a directed bond of G. Let C be the component of $G : (E(G) - |k|)$ which includes the positive ends of the members of $|k|$. Let f be the 0-chain on G such that $f(a) = 1$ or 0 according as a is or is not a vertex of C. By (i) we have

$$\sum \sigma(a) < 0,$$

where the summation is over the vertices of C. If U is the set of all vertices of C in V_1 it follows that U is a subset of V_1 of the kind specified in the enunciation.

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THE NORMALIZER OF CERTAIN MODULAR SUBGROUPS

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Introduction. Let G denote the multiplicative group of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, d are integers and $ad - bc = 1$. G is one of the well-known modular groups. Let $G_0(n)$ denote the subgroup of G characterized by $c \equiv 0 \pmod{n}$, where n is a positive integer. In this note we determine the normalizer of $G_0(n)$ in G, denoted by $\bar{G}_0(n)$. We shall prove the following theorem:

THEOREM 1. *If* $n = 2^\alpha 3^\beta n_0 > 1$, *where* $(n_0, 6) = 1$, *then*

$$\bar{G}_0(n) = G_0(n/2^u 3^v),$$

where $u = \min(3, [\frac{1}{2}\alpha]), v = \min(1, [\frac{1}{2}\beta])$.

Thus in all cases $\bar{G}_0(n) = G_0(n/\Delta)$, where $\Delta|24$. An interesting consequence of this theorem is that if H is a subgroup of G which has $G_0(n)$ for a normal subgroup, then $H = G_0(d)$, where $d|n$ and $(n/d)|24$. This is so since H is included between the groups $G_0(n)$ and $\bar{G}_0(n) = G_0(n/\Delta)$, and so H must be of the form given above by virtue of the theorem quoted in Lemma 1 below.

In addition, Theorem 1 shows that for certain n there are inner automorphisms of $G_0(n)$ arising from elements of G which are not in $G_0(n)$.

We go on now to the proof of Theorem 1. Put

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and note that

$$W^k = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}.$$

LEMMA 1. *If* $n = \sigma^2 Q > 1$ *where* Q *is square-free, then* $\bar{G}_0(n) = G_0(n/\Delta)$, *where* $\Delta|\sigma$.

Proof. The author has shown in **(1)** that if H is a subgroup of G containing $G_0(n)$, then $H = G_0(m)$, where $m|n$. Since $\bar{G}_0(n) \supseteq G_0(n)$, we may put $\bar{G}_0(n) = G_0(m)$, $m|n$. The matrix W^m therefore belongs to $\bar{G}_0(n)$. Since $S \in G_0(n)$ for all n, $W^{-m} S W^m \in G_0(n)$. This implies that $m^2 \equiv 0 \pmod{n}$, or that $(m/\sigma)^2 \equiv 0 \pmod{Q}$, so that $(m/\sigma) \equiv 0 \pmod{Q}$, since Q is square-free. Hence $\sigma Q|m$, and also $m|\sigma^2 Q$. Thus $m = n/\Delta$, where $\Delta|\sigma$.

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LEMMA 2. *Suppose there is some divisor ϵ of σ such that for every element*

$$A = \begin{pmatrix} a & b \\ nc & d \end{pmatrix}$$

of $G_0(n)$, $\epsilon|(d-a)$. Then $\epsilon|\Delta$.

Proof. It is only necessary to show that $W^{n/\epsilon} \in \bar{G}_0(n)$, since then $(n/\Delta)|(n/\epsilon)$, and so $\epsilon|\Delta$. We have

$$W^{-n/\epsilon} A\, W^{n/\epsilon} = \begin{pmatrix} * & * \\ nc + n(d-a)\epsilon^{-1} - nb \cdot n\epsilon^{-2} & * \end{pmatrix}.$$

But $\epsilon^2|n$ (since $\epsilon|\sigma$), and $\epsilon|(d-a)$ by hypothesis. Thus $W^{-n/\epsilon}A\, W^{n/\epsilon} \in G_0(n)$, and so $W^{n/\epsilon} \in \bar{G}_0(n)$.

LEMMA 3. *Suppose $(k, n) = 1$. Then $\Delta|(k^2-1)$.*

Proof. Since $(k, n) = 1$ we can find a, b such that $ak - bn = 1$. The matrix

$$A = \begin{pmatrix} a & b \\ n & k \end{pmatrix}$$

therefore belongs to $G_0(n)$. Since $W^{n/\Delta} \in \bar{G}_0(n)$, $W^{-n/\Delta} A\, W^{n/\Delta} \in G_0(n)$. Performing the multiplications, we see that

$$\frac{n}{\Delta}(k-a) + n\left(1 - \frac{n}{\Delta^2} b\right) \equiv 0 \pmod{n}.$$

This implies Lemma 3, since $\Delta^2|n$ by Lemma 1 and $ak \equiv 1 \pmod{n}$.

LEMMA 4. $\Delta|2^{\alpha+1}\cdot 3$.

Proof. If n is odd, we may choose $k = 2$ in Lemma 3, which implies that $\Delta|3$. If n is even, put $n = 2^\alpha n_1$, where n_1 is odd and $\alpha > 1$. Choose λ so that $\lambda n_1 \equiv -1 \pmod{2^\alpha}$. Then λ is odd. We may choose $k = \lambda n_1 - 2$ in Lemma 3 since

$$\begin{aligned} (k, n) &= (\lambda n_1 - 2, 2^\alpha n_1) \\ &= (\lambda n_1 - 2, 2^\alpha)(\lambda n_1 - 2, n_1) \\ &= 1. \end{aligned}$$

We have

$$\begin{aligned} (k^2-1, n) &= (k^2-1, 2^\alpha n_1) \\ &= (k^2-1, 2^\alpha)(k^2-1, n_1) \\ &= ((\lambda n_1 - 1)(\lambda n_1 - 3), 2^\alpha)((\lambda n_1 - 1)(\lambda n_1 - 3), n_1) \\ &= (8, 2^\alpha)(3, n_1). \end{aligned}$$

But $\Delta|(k^2-1)$, $\Delta|n$ and so $\Delta|(k^2-1, n)$. Taking into account that also $\Delta|\sigma$, we see that $\Delta|2^{\alpha+1}\cdot 3$, and so Lemma 4 is proved.

To complete the proof of Theorem 1, we use Lemma 2 in the following way. Let

$$\begin{pmatrix} a & b \\ nc & d \end{pmatrix}$$

be any element of $G_0(n)$. If $n \equiv 0 \pmod 9$ then $3|\sigma$. Also $ad \equiv 1 \pmod 3$, which implies that $3|(d - a)$. Thus $3|\Delta$. If $n \not\equiv 0 \pmod 9$ then $\sigma \not\equiv 0 \pmod 3$ and so, by Lemma 1, $\Delta \not\equiv 0 \pmod 3$. Hence Δ contains the factor 3 if and only if n is divisible by 9.

If $n \equiv 0 \pmod{64}$ then $8|\sigma$. Also, $ad \equiv 1 \pmod 8$. Thus a and d are odd, and since the square of any odd number is congruent to 1 modulo 8, $8|(d - a)$. Thus $8|\Delta$. Coupled with Lemma 4, we see that Δ contains the factor 8 precisely if and only if n is divisible by 64.

The remaining cases (n divisible by 16 but not by 64, n divisible by 4 but not by 16, n not divisible by 4) are treated similarly.

The proof of Theorem 1 is thus completed.

Reference

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ON UNITARY AND SYMMETRIC MATRICES WITH REAL QUATERNION ELEMENTS

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1. Introduction. In general when symmetric matrices are considered, the elements of the matrix are taken at least in a principal ideal ring. It is interesting to determine what can be attained when the elements are not, in general, commutative and, to this end, the following is concerned with symmetric matrices with elements in the non-commutative field of real quaternions. At the same time some properties of real quaternion unitary matrices are obtained which involve symmetric matrices.

If A has complex elements, a necessary and sufficient condition that A be symmetric is that there exists a unitary matrix U such that $UAU^{T} = D$ is a real diagonal matrix where U^{T} denotes the transpose of U. (See the principal result of **(2)**.) One of the properties of complex matrices importantly involved here is that the transpose of a product of two matrices is the product of their transposes taken in reverse order. For real quaternion matrices this property does not hold in general. The following topics are considered: first, if U is unitary and quaternion, necessary and sufficient conditions that U^{T} be unitary are determined; next, another proof of the above-mentioned theorem for the complex case is given; then, by paralleling this proof, necessary and sufficient conditions are determined that a quaternion matrix have the form UDU^{T} where D is quaternion diagonal and U is real orthogonal (and UDU^{T} is, of course, symmetric); finally, another canonical form for another set of quaternion symmetric matrices is found. (For relevant material on quaternion matrices see **(1)** and **(4)**.)

2. The transpose of a unitary matrix. If U is a unitary matrix (i.e., $UU^{CT} = I$ where U^{CT} denotes the conjugate transpose of U), it does not follow that U^{T} is unitary (as in the complex case). Theorems 1 and 2 supply necessary and sufficient conditions for this, the latter being expressed in terms of symmetric matrices.

THEOREM 1. *If V is a unitary quaternion matrix, a necessary and sufficient condition that V^{T} be unitary is that there exist real orthogonal matrices U and W such that $UVW = D$ is a diagonal quaternion matrix.*

Proof. Let $V = V_1 + jV_2$, where V_1 and V_2 are complex matrices, and let $V_1 = T_1 + iT_2$ and $V_2 = W_1 + iW_2$ where T_1, T_2, W_1, and W_2 are real matrices.

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Let V^T be unitary. Since $VV^{CT} = V^{CT}V = I$, the following relations result:

$$V_1^{CT}V_1 + V_2^{CT}V_2 = I, \quad V_1V_1^{CT} + V_2^{C}V_2^{T} = I,$$
$$V_1^{T}V_2 = V_2^{T}V_1, \quad V_2V_1^{CT} = V_1^{C}V_2^{T}.$$

Similarly, since $V^TV^C = V^CV^T = I$,[1]

$$V_1^{CT}V_1 + V_2^{T}V_2^{C} = I, \quad V_1V_1^{CT} + V_2V_2^{CT} = I,$$
$$V_2^{T}V_1^{C} = V_1^{CT}V_2, \quad V_2V_1^{T} = V_1V_2^{T}.$$

Since $V_1^{CT}V_1 + V_2^{CT}V_2 = V_1^{CT}V_1 + V_2^{T}V_2^{C}$ and $V_1V_1^{CT} + V_2^{C}V_2^{T} = V_1V_1^{CT} + V_2V_2^{CT}$, it follows that $V_2^{CT}V_2 = V_2^{T}V_2^{C}$ and $V_2^{C}V_2^{T} = V_2V_2^{CT}$. In a similar manner, $V_1^{CT}V_1 = V_1^{T}V_1^{C}$ and $V_1V_1^{CT} = V_1^{C}V_1^{T}$ since $I = I^T$. Consider the following relations:

$$V_2^{T}V_1 = V_1^{T}V_2, \quad V_2^{T}V_1^{C} = V_1^{CT}V_2, \quad V_2V_1^{T} = V_1V_2^{T}, \quad V_2V_1^{CT} = V_1^{C}V_2^{T},$$
$$V_2^{CT}V_2 = V_2^{T}V_2^{C}, \quad V_2V_2^{CT} = V_2^{C}V_2^{T}, \quad V_1^{CT}V_1 = V_1^{T}V_1^{C}, \quad V_1V_1^{CT} = V_1^{C}V_1^{T}.$$

Since $V_1 = T_1 + iT_2$ and $V_2 = W_1 + iW_2$, it follows from the first pair of relations that $W_iT_j = T_jW_i$, for $i = 1, 2$; $j = 1, 2$. From the next pair it follows that $W_iT_j^T = T_jW_i^T$, for $i = 1, 2$; $j = 1, 2$. From the next pair it follows that $W_1^TW_2 = W_2^TW_1$ and $W_2W_1^T = W_1W_2^T$, and from the last pair that $T_1^TT_2 = T_2^TT_1$ and $T_2T_1^T = T_1T_2^T$. Therefore the set of real matrices $\{T_1, T_2, W_1, W_2\}$ is such that if X and Y are any two matrices of the set, then XY^T and Y^TX are real symmetric. Now the following holds by a known theorem **(3)**: if A_i is an arbitrary set of non-zero complex matrices, there exist unitary matrices U and W such that $UA_iW = D_i$ where D_i is diagonal and real if and only if $A_iA_j^{CT} = A_jA_i^{CT}$ and $A_j^{CT}A_i = A_i^{CT}A_j$ for all i and j. In our case the matrices are all real and it is easily seen that the U and W will be real orthogonal matrices. Then since $V = T_1 + iT_2 + j(W_1 + iW_2)$, therefore $UVW = UT_1W + iUT_2W + j(UW_1W + iUW_2W) = D_1 + iD_2 + j(D_3 + iD_4)$ is a diagonal quaternion matrix.

Conversely, let V be unitary such that real orthogonal matrices U and W exist so that $UVW = D$ is quaternion and diagonal. Then $V = U^TDW^T$, and since U^T and W^T are real, it is true that $V^T = (U^TDW^T)^T = WDU$ and so $V^T \cdot (V^T)^{CT} = (WDU)(WDU)^{CT} = WDU \cdot U^TD^CW^T = I$, since $D \cdot D^{CT} = UVW \cdot W^TV^{CT}U^T = I$, and so V^T is unitary.

COROLLARY 1.1. *If V is a complex unitary matrix, there exist real orthogonal matrices U and W such that $UVW = D$ is a diagonal matrix with complex elements.*

This follows since in this case V^T is always unitary.

Let us define a unitary quaternion matrix U to be T-unitary if U^T is unitary.

COROLLARY 1.2. *A* T*-unitary matrix V is symmetric if and only if there exists a real orthogonal matrix U such that $UVU^{T} = D$ is a quaternion diagonal matrix.*

For if V is T-unitary and symmetric, then in the above proof $T_i = T_i^{T}$, $W_i = W_i^{T}$ for $i = 1, 2$, and the set of matrices $\{T_1, T_2, W_1, W_2\}$ are real, symmetric, and commutative in pairs and consequently can be diagonalized by a single real orthogonal similarity transformation. The converse is evident.

The following may also be noted:

THEOREM 2. *If V is unitary, then V^{T} is unitary if and only if VV^{T} and $V^{T}V$ are symmetric.*

If V and V^{T} are unitary, by the preceding theorem there exist real orthogonal matrices U and W such that $V = UDW$ where D is quaternion and diagonal. Then $V^{T} = W^{T}DU^{T}$ and $VV^{T} = UD^{2}U^{T}$ and $V^{T}V = W^{T}D^{2}W$ are symmetric.

Conversely, let V be unitary and let $V^{T}V$ and VV^{T} be symmetric. Let $V = V_1 + jV_2$ where V_1 and V_2 are complex; then

$$\begin{aligned} VV^{T} &= V_1V_1^{T} - V_2^{C}V_2^{T} + j(V_2V_1^{T} + V_1^{C}V_2^{T}) \\ &= V_1V_1^{T} - V_2V_2^{CT} + j(V_1V_2^{T} + V_2V_1^{CT}) \\ V^{T}V &= V_1^{T}V_1 - V_2^{CT}V_2 + j(V_2^{T}V_1 + V_1^{CT}V_2) \\ &= V_1^{T}V_1 - V_2^{T}V_2^{C} + j(V_1^{T}V_2 + V_2^{T}V_1^{C}). \end{aligned}$$

From this it follows that

$$V_2^{C}V_2^{T} = V_2V_2^{CT},\ V_2^{T}V_2^{C} = V_2^{CT}V_2,\ V_2V_1^{T} + V_1^{C}V_2^{T} = V_1V_2^{T} + V_2V_1^{CT},$$
$$V_2^{T}V_1 + V_1^{CT}V_2 = V_1^{T}V_2 + V_2^{T}V_1^{C}.$$

The latter two relations may be rewritten as

$$\begin{aligned} V_1^{C}V_2^{T} - V_2V_1^{CT} &= V_1V_2^{T} - V_2V_1^{T}, \\ V_2^{T}V_1 - V_1^{T}V_2 &= V_2^{T}V_1^{C} - V_1^{CT}V_2. \end{aligned}$$

Now $VV^{CT} = I = V^{CT}V$ and so:

$$\begin{aligned} I &= V_1V_1^{CT} + V_2^{C}V_2^{T} + j(V_2V_1^{CT} - V_1^{C}V_2^{T}) \\ &= V_1^{CT}V_1 + V_2^{CT}V_2 + j(-V_2^{T}V_1 + V_1^{T}V_2), \\ I^{T} &= V_1^{C}V_1^{T} + V_2V_2^{CT} + j(V_1^{C}V_2^{T} - V_2V_1^{CT}) \\ &= V_1^{T}V_1^{C} + V_2^{T}V_2^{C} + j(-V_1^{T}V_2 + V_2^{T}V_1). \end{aligned}$$

This means that

$$\begin{aligned} V_1^{C}V_1^{T} + V_2V_2^{CT} &= I = V_1^{T}V_1^{C} + V_2^{T}V_2^{C}, \\ V_1^{C}V_2^{T} - V_2V_1^{CT} &= 0 = -V_1^{T}V_2 + V_2^{T}V_1. \end{aligned}$$

Substituting the relations obtained above in these expressions, it follows that

$$V_1^C V_1^T + V_2^C V_2^T = I = V_1^T V_1^C + V_2^{CT} V_2,$$
$$V_1 V_2^T - V_2 V_1^T = 0 = V_2^T V_1^C - V_1^{CT} V_2.$$

But this means that $V^C V^T = I = V^T V^C$ and so V^T is unitary.

It may be noted that if the matrices in the above Theorem are all complex, the result holds since then V^T is always unitary and VV^T and $V^T V$ are always symmetric.

It is known **(1)** that a matrix A is normal if and only if there exists a unitary matrix U such that $UAU^{CT} = D$ is a complex diagonal matrix. It is of interest to determine what characterizes normal matrices which can be brought into diagonal form by T-unitary matrices.

THEOREM 3. *A normal quaternion matrix A can be brought into complex diagonal form by means of a* T*-unitary similarity transformation if and only if A is unitarily similar to a complex symmetric matrix S (i.e., $U_1 A U_1^{CT} = S$) under a matrix of the form $U_1 = DW_1$ where W_1 is real orthogonal and D is a unitary quaternion diagonal matrix.*

If $A = U^{CT} D_1 U$ where U is T-unitary and D_1 is diagonal and complex, there exist real orthogonal matrices V and W such that $VUW = D$ is a quaternion diagonal unitary matrix. Then

$$W^T A W = W^T U^{CT} V^T V D_1 V^T V U W = D^{CT} S D$$

where $S = VD_1V^T$ is complex symmetric and normal. Therefore $DW^T AWD^{CT} = S$ where $U_1 = DW^T$ is unitary and of the above form.

If there exists a $U_1 = DW^T$ of the type described such that $U_1 A U_1^{CT} = S$ is complex symmetric and normal, then there exists a real orthogonal matrix V such that $V^T S V = D_1$ and so

$$A = WD^{CT} V D_1 V^T D W^T = U^{CT} D_1 U,$$

where U is T-unitary.

COROLLARY 3.1. *If $A = A^{CT}$, S is real symmetric.*

COROLLARY 3.2. *If $A = -A^{CT}$, $S = iT$, where T is real symmetric.*

COROLLARY 3.3. *If $A \cdot A^{CT} = I$, S is unitary symmetric.*

3. Matrices of the form UDU^T, U real orthogonal and D quaternion and diagonal. Let us consider first the following proof of the above-mentioned theorem:

THEOREM 4. *If $A = A^T$ has complex elements, there exists a complex unitary matrix U such that $UAU^T = D$ is a real diagonal matrix.* (The converse is obvious.)

Let $A = HV = VK$ (where H, K are hermitian and V is unitary) be the polar form of A. (If A is non-singular, H, V, and K are uniquely determined **(6)**; if A is singular **(5)**, some arbitrariness is involved in V.) Then $A = HV = VK = V^{\mathrm{T}}H^{\mathrm{T}} = K^{\mathrm{T}}V^{\mathrm{T}}$ and so $H = K^{\mathrm{T}}$. If A is non-singular, $V = V^{\mathrm{T}}$; if A is singular, this can also be attained by means of a proper choice of V as the following will show.

Let $UHU^{\mathrm{CT}} = D = D_1 \dot{+} 0$ where D_1 is diagonal and real with like roots arranged together along the diagonal. Let $UVU^{\mathrm{CT}} = W$, $UV^{\mathrm{T}}U^{\mathrm{CT}} = W_1$, and $UKU^{\mathrm{CT}} = UH^{\mathrm{T}}U^{\mathrm{CT}} = M$. Then $H^{\mathrm{T}} = U^{\mathrm{T}}DU^{\mathrm{C}}$ and $UH^{\mathrm{T}}U^{\mathrm{CT}} = UU^{\mathrm{T}}DU^{\mathrm{C}}U^{\mathrm{CT}} = M$. Let D_1 be of order r, i.e., $r < n$ where n is the order of D. Then

$$UAU^{\mathrm{CT}} = DW = DW_1 = WM = W_1M$$

and so $$WUU^{\mathrm{T}}DU^{\mathrm{C}}U^{\mathrm{CT}} = W_1UU^{\mathrm{T}}DU^{\mathrm{C}}U^{\mathrm{CT}}, \qquad WUU^{\mathrm{T}}D = W_1UU^{\mathrm{T}}D.$$

From $DW = DW_1$, it follows that W and W_1 have like first r rows. From $WUU^{\mathrm{T}}D = W_1UU^{\mathrm{T}}D$, WUU^{T} and W_1UU^{T} have like first r columns. Since W and W_1 have like first r rows, WUU^{T} and W_1UU^{T} have like first r rows also. Since $DW = W_1M = W_1UU^{\mathrm{T}}DU^{\mathrm{C}}U^{\mathrm{CT}}$, then $DWUU^{\mathrm{T}} = W_1UU^{\mathrm{T}}D$ Then

$$WUU^{\mathrm{T}} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & X \end{bmatrix}$$

where B_{11} is an $r \times r$ matrix and W_1UU^{T} has the same form except for the matrix X. Let the elements common to both matrices (i.e., the elements of B_{11}, B_{12} and B_{21}) be denoted by w_{ij} (according to their row and column location in WUU^{T}). From $DWUU^{\mathrm{T}} = W_1UU^{\mathrm{T}}D$, it follows that if d_j denotes the jth diagonal element of D,

$$d_iw_{ij} = 0, \quad (i = 1, 2, \ldots, r;\ j = r + 1, \ldots, n),$$
$$w_{ij}d_j = 0, \quad (j = 1, 2, \ldots, r;\ i = r + 1, \ldots, n).$$

For this range of subscripts, then, $w_{ij} = 0$, i.e., B_{12} and B_{21} are zero matrices. Therefore

$$WUU^{\mathrm{T}} = B_{11} \dot{+} X, \qquad W_1UU^{\mathrm{T}} = B_{11} \dot{+} Y,$$
$$W = (B_{11} \dot{+} X)U^{\mathrm{C}}U^{\mathrm{CT}} \qquad W_1 = (B_{11} \dot{+} Y)U^{\mathrm{C}}U^{\mathrm{CT}}$$
$$= UVU^{\mathrm{CT}}, \qquad = UV^{\mathrm{T}}U^{\mathrm{CT}}.$$

Therefore

$$V = U^{\mathrm{CT}}\begin{bmatrix} B_{11} & 0 \\ 0 & X \end{bmatrix}U^{\mathrm{C}}, \quad V^{\mathrm{T}} = U^{\mathrm{CT}}\begin{bmatrix} B_{11} & 0 \\ 0 & Y \end{bmatrix}U^{\mathrm{C}}.$$

But

$$V^{\mathrm{T}} = U^{\mathrm{CT}}\begin{bmatrix} B_{11}^{\mathrm{T}} & 0 \\ 0 & X^{\mathrm{T}} \end{bmatrix}U^{\mathrm{C}} = U^{\mathrm{CT}}\begin{bmatrix} B_{11} & 0 \\ 0 & Y \end{bmatrix}U^{\mathrm{C}},$$

and so $B_{11}^{\mathrm{T}} = B_{11}$, $Y = X^{\mathrm{T}}$. Now if X is chosen to be any unitary and symmetric matrix of dimension $(n - r) \times (n - r)$ then $X = X^{\mathrm{T}}$ and $V^{\mathrm{T}} = V$.

Consequently, it is always possible in the singular case to find a $V = V^T$ which is suitable for the polar unitary matrix of A.

Now $A = HV = VH^T$ and so

$$UAU^T = UHU^{CT}UVU^T = UVU^TU^CH^TU^T = DW = WD$$

where $W = UVU^T = W^T$ is unitary. If $W = W_3 + iW_4$ where W_3 and W_4 are real, it is then true that D, W_3, and W_4 are real symmetric matrices which commute in pairs and so there exists a real orthogonal matrix U_1 that diagonalizes all of them. Therefore $U_1UAU^TU_1^T = U_1(DW)U_1^T = D_2$ where D_2 is diagonal with complex elements along the diagonal. Let the diagonal elements of D_2 be

$$\rho_k e^{-i\theta_k} \qquad (k = 1, 2, \ldots, r),$$

and form the diagonal unitary matrix D_u with

$$e^{k\theta_k} \qquad (k = 1, 2, \ldots, r),$$

in the first r diagonal positions followed by ones in the remaining. Then $(D_uU_1U)\,A\,(D_uU_1U)^T = D_3$ is real and diagonal and the Theorem is true.

Next consider the case where $A = A^T$ is a quaternion matrix and the possibility of transforming such a matrix by means of a real orthogonal matrix into a diagonal quaternion matrix. It may be noted first that a quaternion matrix has a polar form **(4)** $A = HV = VK$ with the same properties as mentioned above in connection with the singular and non-singular case. V may, of course, be merely unitary and not necessarily T-unitary.

THEOREM 5. *If A is a quaternion matrix, there exists a real orthogonal matrix U such that $UAU^T = D$ is diagonal and quaternion if and only if A is symmetric with a real hermitian polar matrix.*

Let $A = HV = VK$ where H has real elements and A is symmetric. Then $A^T = (HV)^T = V^TH^T = V^TH$ and so $VK = V^TH$. Whether A is singular or non-singular, the preceding proof may be followed; the matrix U is real in this case, and the relations $DW = DW_1 = WD = W_1D$ result so that $W = B_{11} \dotplus X$ and $W_1 = B_{11} \dotplus Y$ from which $V = U^T(B_{11} \dotplus X)U$ and $V^T = U^T(B_{11} \dotplus Y)U$. Then $V^T = U^T(B_{11}{}^T \dotplus X^T)U = U^T(B_{11} \dotplus Y)U$, which is permissible since U is real. Therefore $B_{11}{}^T = B_{11}$ and $X^T = Y$. If A is singular and if X is chosen to be unitary and symmetric, then $V = V^T$; if A is non-singular, then X does not appear above and $V = V^T$ holds automatically. Then $HV = VH = A$ where V is T-unitary and symmetric. By Corollary 1.2, there exists a real orthogonal U_1 such that $U_1VU_1^T = D_1$ is diagonal and quaternion with like diagonal elements grouped together. Then

$$U_1AU_1^T = U_1VU_1^TU_1HU_1^T = U_1HU_1^TU_1VU_1^T = D_1U_1HU_1^T = U_1HU_1^TD_1.$$

Then $U_1HU_1^T$ falls into a direct sum of real symmetric block matrices as

determined by D_1 and a real orthogonal U_2 can then be determined so that $U_2U_1AU_1^TU_2^T = D$ is quaternion diagonal.

Conversely, if $A = U^TDU$ for U real orthogonal, let $D = D_r \cdot D_q$ where D_r consists of the real parts of each diagonal element of D and D_q is composed of the corresponding quaternion part of absolute value one. (If A is singular, D_q is arbitrary to some extent but not necessarily diagonal.) Then

$$A = U^TD_rU \cdot U^TD_qU = U^TD_qU \cdot U^TD_rU$$

is symmetric and the hermitian polar matrix, U^TD_rU, has the required property.

THEOREM 6. *If A is a symmetric quaternion matrix with a real polar unitary matrix, there exists a real orthogonal matrix U such that*

$$UAU^T = \begin{bmatrix} D_1 & M \\ M^T & D_2 \end{bmatrix},$$

where D_1 and D_2 are real diagonal and $M = -M^C$; and conversely.

Let $A = HV = VK = V^TH^T = K^TV^T$ since V is real orthogonal. As before $H = K^T$ and $V = V^T$. (In the singular case, let U be a complex unitary matrix such that $UVU^{CT} = D$; then $UV^TU^{CT} = D^{CT}$ and applying this transformation to $V^TH^T = VH^T = HV = HV^T$, there results

$$D^{CT}M = DM = M_1D = M_1D^{CT}$$

where $M = UH^TU^{CT}$ and $M_1 = UHU^{CT}$ are hermitian, and D has along the diagonal elements of absolute value one.) If in a given row, say k, of M one element $m_{kj} \neq 0$, then $\bar{d}_k m_{kj} = d_k m_{kj}$ (where d_k is the kth diagonal element of D) and so d_k is $+1$ or -1; if all elements of the kth row are zero, d_k is arbitrary except that it must have absolute value one. If the kth row of M is zero, so is the corresponding row of M_1, and conversely. By choosing the arbitrary d_k to be either $+1$ or -1, it is evident that $D = D^{CT}$ and therefore $V = V^T$.

If $V = I$, (or can be taken to be I), then $A = H = H^{CT} = H^T$ and so A is real and symmetric and consequently can be brought into diagonal form by a real orthogonal transformation. For the case where A is quaternion symmetric (as considered here), there exists a real orthogonal matrix W so that $WVW^T = I_1 \dotplus I_2 = D$ where I_1 and I_2 are diagonal with $+1$ and -1, respectively, along the diagonal. Then

$$WAW^T = WHW^TWVW^T = WVW^TWH^TW^T = H_1D = DH_1^T$$

where $H_1 = WHW^T$. Let

$$H_1 = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix}$$

be subdivided to correspond to $I_1 \dot{+} I_2$ (where $H_1^T = H_1^C$). Then $K_1 = K_1^T$ and $K_4 = K_4^T$ are real symmetric and $K_2 = -K_2^C$ and $K_3 = -K_3^C$ while $K_3 = -K_2^T$. Therefore,

$$WAW^T = \begin{bmatrix} K_1 & -K_3 \\ -K_2^T & -K_4 \end{bmatrix}.$$

Let $W_1K_1W_1^T = D_1$ and $W_2K_4W_2^T = D_2$ be real and diagonal where W_1 and W_2 are real orthogonal, and $U_1 = W_1 \dot{+} W_2$. Then

$$U_1WAW^TU_1^T = \begin{bmatrix} D_1 & M \\ M^T & D_2 \end{bmatrix},$$

where $M = -W_1K_2W_2^T = -M$. The converse is evident.

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SUBMETHODS OF REGULAR MATRIX SUMMABILITY METHODS

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1. Introduction By a submethod of a regular matrix method A we mean a method (see 1 or 3) whose matrix is obtained by deleting a set of rows from the matrix A. We establish a one-one correspondence between the submethods of A and the points of the interval $0 < \xi \leqslant 1$. We designate the submethod which corresponds to ξ by $A(\xi)$ and are accordingly able to speak of sets of submethods of measure 0, of the first category, etc. Now, every bounded sequence $\{s_n\}$ is summed by certain submethods of A. We find that if $\{s_n\}$ is not summed by A itself, then the set of submethods of A by means of which it is summed is of the first category, but may be either of measure 0 or 1. A submethod of A may be either equivalent to A or strictly stronger than A. We find that the set of submethods equivalent to A is always of the first category. On the other hand, every regular method A has equivalent methods B and C such that the set of submethods of B which are equivalent to B is of measure 0 and the set of submethods of C which are equivalent to C is of measure 1. However, certain important methods are equivalent to almost all of their submethods, but we prove this only for the $(C, 1)$ method. We consider only bounded sequences, so that equivalence, etc., are relative to the set of bounded sequences. There is some analogy between this work and work on the Borel property (4; 5).

2. Category. Let $A = (a_{mn})$ be an infinite matrix. We establish a one-one correspondence between the submethods of A and the points of the interval $0 < \xi \leqslant 1$ by associating with each point ξ in this interval the submatrix of A whose nth row is deleted if and only if $a_n = 0$ in the non-terminating binary expansion $.a_1a_2 \ldots a_n \ldots$ of ξ. We designate the submatrix corresponding to ξ as $A(\xi)$ and use the same notation for the corresponding summability method. We say that a set of submethods $A(\xi)$, $\xi \in E$, has a specific property whenever the set E has this property. We shall refer only to regular methods although it will be clear that our results hold for other methods as well.

THEOREM I. *If A is a regular method and $\{s_n\}$ is a bounded sequence which is not summable by means of A, then the set of submethods of A by means of which $\{s_n\}$ is summable is of the first category.*

Proof. Let ξ_0 be a real number for which $A(\xi_0)$ does not sum $\{s_n\}$, and let D be the set of ξ obtained by changing the binary expansion of ξ_0 in a finite

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number of places. D is everywhere dense, and there is a $k > 0$ such that, for every $\xi \in D$, the $A(\xi)$ transform $\{t_n, \xi\}$ of $\{s_n\}$ satisfies the condition

$$\limsup_{n\to\infty} t_{n,\xi} > \liminf_{n\to\infty} t_{n,\xi} + k.$$

We now consider the set S_n of all ξ such that there are $\mu, \nu > n$ for which $|t_{\nu,\xi} - t_{\mu,\xi}| > k$. For every n, the set S_n is open. For, if $\xi \in S_n$, $\mu > \nu > n$, and $|t_{\nu,\xi} - t_{\mu,\xi}| > k$, and if $|\eta - \xi| < 2^{-\nu-1}$, then $|t_{\nu,\eta} - t_{\mu,\eta}| = |t_{\nu,\xi} - t_{\mu,\xi}| > k$, so that $\eta \in S_n$. But $D \subset S_n$, for every n, so that the set

$$S = \bigcap_{n=1}^{\infty} S_n$$

is an everywhere dense set of type G_δ. Hence, its complement is of the first category. Finally, it is evident that for every $\xi \in S$ the sequence $\{s_n\}$ is not summable by means of $A(\xi)$.

Although the set of those $A(\xi)$ which sum $\{s_n\}$ is of the first category, it is non-denumerable. For, if $\{s_n\}$ is summable by means of $A(\xi)$ then it is summable by means of every submethod of $A(\xi)$.

We now show that the set of submethods of a regular method A which are equivalent to A is of the first category.

LEMMA 1. *Let A be a regular method for which*

$$\lim_{m\to\infty} \max_n |a_{mn}| = 0.$$

There is a strictly increasing $F(n)$ such that if $\{s_n\}$ is A summable and $s_n = 1$, $n = n_\nu$ $(\nu = 1, 2, \ldots)$ and $s_n = 0$ for all other n, where $n_{\nu+1} - n_\nu > F(n_\nu)$ for an infinite number of values of ν, then

$$A - \lim_{n\to\infty} s_n = 0.$$

Proof. For each n, there is an $r > n$ and an $F(n)$ such that

$$\sum_{\mu=1}^{\bar{n}} |a_{r\mu}| < \frac{1}{2n}, \qquad \sum_{\mu=\bar{n}+F(\bar{n})}^{\infty} |a_{r\mu}| < \frac{1}{2n}. \tag{1}$$

Let $\{s_n\}$ satisfy the conditions of the Lemma, and let ν be such that $n_{\nu+1} - n_\nu > F(n_\nu)$. Let $r > n_\nu$ satisfy (1) for n_ν. (We write $\bar{n}$ for n_ν.)

$$\begin{aligned} t_r &= \sum_{\mu=1}^{\infty} a_{r\mu} s_\mu \\ &= \sum_{\mu=1}^{\bar{n}} a_{r\mu} s_\mu + \sum_{\mu=\bar{n}+1}^{\bar{n}+F(\bar{n})-1} a_{r\mu} s_\mu + \sum_{\mu=\bar{n}+F(\bar{n})}^{\infty} a_{r\mu} s_\mu \\ &= \sum_{\mu=1}^{\bar{n}} a_{r\mu} s_\mu + \sum_{\mu=\bar{n}+F(\bar{n})}^{\infty} a_{r\mu} s_\mu, \end{aligned}$$

and

$$|t_r| < \sum_{\mu=1}^{\bar{n}} |a_{r\mu}| + \sum_{\mu=\bar{n}+F(\bar{n})}^{\infty} |a_{r\mu}| < \frac{1}{\bar{n}}.$$

Hence, if the A transform $\{t_n\}$ of $\{s_n\}$ converges, its limit must be 0.

For a detailed discussion of counting functions used in a different way see **(6)**.

LEMMA 2. *With the same restrictions on A as in Lemma 1, there is a $G(n)$ such that if the binary expansion of ξ is 1 for $n = n_\nu$ $(\nu = 1, 2, \ldots)$ and is 0 everywhere else, and if $n_{\nu+1} - n_\nu > G(n_\nu)$ for an infinite number of values of ν, then $A(\xi)$ is strictly stronger than A.*

Proof. We need only choose the sequence $\{G(n)\}$ so that whenever $r - n > G(n)$ there is a $\nu = \nu(n)$ such that

$$\sum_{m=\nu}^{\infty} |a_{nm}| < \frac{1}{n} \text{ and } \sum_{m=1}^{\nu+F(n)} |a_{rm}| < \frac{1}{n}.$$

for every n, where $\nu(n)$ is strictly increasing. Suppose ξ satisfies the condition of the Lemma. Let

$$n_1 < m_1 \leqslant n_2 < m_2 \leqslant \ldots \leqslant n_k < m_k \leqslant \ldots$$

be places for which the binary expansion of ξ is 1, such that for every k, $m_k - n_k > G(n_k)$, and the binary expansion of ξ is 0 at all places between n_k and m_k. Supposing that s_n has been defined for all $n < \nu(n_k)$, we let $s_n = 0$ for

$$\nu(n_k) \leqslant n \leqslant \nu(n_k) + F(n_k)$$

and $s_n = 1$ for

$$\nu(n_k) + F(n_k) < n < \nu(n_{k+1}).$$

The construction of the sequence $\{s_n\}$ is then completed by induction. It is evidently summable to 1 by the $A(\xi)$ method. If $\{s_n\}$ were summable by the A method then, by Lemma 1, its limit would be 0. Hence $\{s_n\}$ is not A summable.

LEMMA 3. *If A is a regular row finite method for which*

$$\limsup_{m \to \infty} \max_{n} |a_{mn}| > 0$$

there is also a $G(n)$ for which the conclusion in Lemma 2 holds.

Proof. By hypothesis, there is a sequence m_ν $(\nu = 1, 2, \ldots)$ and a $k > 0$ such that, for every ν, there is a k_ν for which

$$|a_{m_\nu k_\nu}| > k.$$

Evidently,

$$\lim_{\nu \to \infty} k_\nu = \infty.$$

We define $G(n)$ so that whenever $n_{\nu+1} - n_\nu > G(n_\nu)$, it follows that there is an m_μ with

$$n_\nu < m_\mu < n_{\nu+1}, \quad |a_{n_{\nu+1} k_\mu}| < \frac{1}{n_\nu}, \quad k_\mu > k(n_\nu),$$

where $k(m)$ is defined so that $|a_{m,k(m)}|$ is the last non-zero element of the mth row. Now, if ξ satisfies the conditions of the Lemma, we may define $\{s_n\}$ to be 1 at a certain infinite set of values of k_r and 0 everywhere else such that $\{s_n\}$ is $A(\xi)$ summable to 0. But this $\{s_n\}$ is not A summable.

LEMMA 4. *For every regular A, there is a regular row finite B such that $A(\xi)$ is equivalent to $B(\xi)$ for all $0 < \xi \leqslant 1$.*

Proof. For every n, there is an m_n such that

$$\sum_{m=m_n+1}^{\infty} |a_{nm}| < \frac{1}{n}.$$

We let $b_{nm} = a_{nm}$, if $m \leqslant m_n$, and $b_{nm} = 0$ if $m > m_n$. The matrix $B = (b_{nm})$ has the required character.

We now prove:

THEOREM II. *For every regular method A, the set of ξ for which $A(\xi)$ is equivalent to A is of the first category.*

Proof. Because of Lemma 4, we need only prove the theorem for row finite methods. Suppose then that A is row finite. Let E_p be the set of all ξ such that

$$n_{r+1}(\xi) - n_r(\xi) < G\{n_r(\xi)\}$$

for all $n_r(\xi) \geqslant p$, where $n_1(\xi), n_2(\xi), \ldots$ are the places at which the binary expansion of ξ is 1. We show that E_p is a closed, nowhere dense set. For, if $\eta \notin E_p$, there is an $n_\mu > p$ for which

$$n_{\mu+1}(\eta) - n_\mu(\eta) > G\{n_\mu(\eta)\}$$

so that η is at a positive distance from E_p. Hence, the complement $C(E_p)$ of E_p is open. That $C(E_p)$ is everywhere dense is obvious since a point η can have arbitrary 0's and 1's in its first n places, for every n, and belong to $C(E_p)$. It follows that the set

$$E = \bigcup_{p=1}^{\infty} E_p$$

is of the first category. But, by Lemmas 2 and 3, this set contains all ξ for which $A(\xi)$ is equivalent to A.

3. Measure. The situation is not as clear cut with respect to measure as it is with respect to category. Indeed, we have:

THEOREM III. *For every regular method A there exist methods B and C, equivalent to A, such that B is equivalent to $B(\xi)$ for almost all values of ξ and $C(\xi)$ is strictly stronger than C for almost all values of ξ.*

Proof. Let a_n, $(n = 1, 2, \ldots)$ be the rows of A, and let B be the matrix whose rows are $a_1, a_2, a_2, \ldots, a_n, \ldots, a_n, \ldots$ where a_n is repeated 2^{n-1} times

for every n. Let E_n be the set of ξ for which a_n is a row of $B(\xi)$. The measure of E_n is $1-2^{1-n}$. Let S be the set of ξ for which $B(\xi)$ contains all but a finite number of the rows of A. Then

$$S = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} S_n$$

is of measure 1. This little argument is sometimes called the Borel-Cantelli lemma (2, p. 201). Obviously, $B(\xi)$ is equivalent to B for every $\xi \in S$, and B is equivalent to A.

Now, let D be strictly stronger than A and let C be the matrix whose rows are $a_1, d_1, a_2, d_2, d_2, \ldots, a_n, d_n, \ldots, d_n, \ldots$ where the nth row, d_n, of D is repeated 2^{n-1} times. The method C is then equivalent to A. But for almost all values of ξ, all but a finite number of rows of $C(\xi)$ are taken from D. Hence, $C(\xi)$ is strictly stronger than C for almost all values of ξ.

We show next that if A is the $(C, 1)$ method, then $A(\xi)$ is equivalent to A for almost all ξ.

LEMMA 5. *There is an integer valued function $\phi(n)$ such that*

$$\lim_{n\to\infty} \frac{\phi(n)}{n} = 0, \qquad \sum_{n=1}^{\infty} 2^{-\phi(n)} < \infty.$$

We omit the proof which can be easily supplied by the reader.

We consider the one-one correspondence between the set of increasing sequences of positive integers and the set of points in the interval $0 < x \leqslant 1$, obtained by mating each sequence

$$n_1 < n_2 < \ldots < n_k < \ldots$$

with the point whose non-terminating binary expansion $.a_1 a_2 \ldots a_n \ldots$ has $a_n = 1$ for $n = n_k$ $(k = 1, 2, \ldots)$ and $a_n = 0$ everywhere else. The measure of a set of increasing sequences is defined as the measure of its set of images in the interval $0 < x \leqslant 1$.

LEMMA 6. *The set of increasing sequences $n_1 < n_2 < \ldots < n_k < \ldots$ of positive integers which satisfy the condition*

$$\lim_{k\to\infty} \frac{n_{k+1} - n_k}{n_k} = 0$$

is of measure 1.

Proof. For every k, the measure of the set for which

$$n_{k+1} - n_k > \phi(k)$$

is $2^{-\phi(k)}$. Thus, for every m, the measure of the set for which there is at least one $k > m$ for which $n_{k+1} - n_k > \phi(k)$ does not exceed

$$\sum_{k=m}^{\infty} 2^{-\phi(k)}.$$

It follows, by Lemma 5, that the set for which $n_{k+1}-n_k \leqslant \phi(k)$ for all but a finite number of values of k is of measure 1. But $n_{k+1}-n_k \leqslant \phi(k)$ implies

$$\frac{n_{k+1}-n_k}{n_k} \leqslant \frac{\phi(k)}{n_k} \leqslant \frac{\phi(k)}{k}.$$

Since

$$\lim_{k\to\infty} \frac{\phi(k)}{k} = 0,$$

it follows that the set of increasing sequences for which

$$\lim_{k\to\infty} \frac{n_{k+1}-n_k}{n_k} = 0$$

has measure 1.

Let $\xi \in (0, 1)$, and let $n_1 < n_2 < \ldots < n_k < \ldots$ be the sequence of integers at which 1 appears in its binary expansion.

LEMMA 7. *If A is the $(C, 1)$ method and ξ is such that*

$$\lim_{k\to\infty} \frac{n_{k+1}-n_k}{n_k} = 0$$

then $A(\xi)$ is equivalent to A.

Proof. Let $\{s_n\}$ be a bounded sequence, $|s_n| < M$, for all n, and let $\{t_n\}$ be the A transform of $\{s_n\}$. Then, for every n and k, we have

$$\begin{aligned} |t_n - t_{n+k}| &= \left|\frac{1}{n}\sum_{i=1}^{n} s_i - \frac{1}{n+k}\sum_{i=1}^{n+k} s_i\right| \\ &< \left(\frac{1}{n} - \frac{1}{n+k}\right)\sum_{i=1}^{n} |s_i| + \frac{1}{n+k}\sum_{i=n+1}^{n+k} |s_i| < \frac{2kM}{n+k}. \end{aligned}$$

Suppose $\{s_n\}$ is summable by the $A(\xi)$ method. Let $\epsilon > 0$. There is a k such that, for every $j > k$, $|t_{n_k}-t_{n_j}| < \frac{1}{2}\epsilon$ and $n_j \leqslant n < n_{j+1}$ implies

$$|t_{n_j} - t_n| < \frac{2(n-n_j)}{n} M < \frac{2(n_{j+1}-n_j)}{n_j} M < \tfrac{1}{2}\epsilon.$$

Hence, $|t_n-t_{n_k}| < \epsilon$, for every $n > n_k$, and so $\{s_n\}$ is summable by means of A.

By Lemmas 6 and 7, we have:

THEOREM IV. *The $(C, 1)$ summability method is equivalent to almost all of its submethods.*

Finally, we prove:

THEOREM V. *If A is a regular method, and $\{s_n\}$ is a bounded sequence not summable by means of A, then the set of submethods of A which sums $\{s_n\}$ is of measure either 0 or 1, and either value can occur.*

Proof. Let $\{s_n\}$ be a bounded sequence summed by A_1 to 1 and by A_2 to 0. Form A by intertwining the rows of A_1 with those of A_2 so that almost all

submethods of A are equivalent to submethods of A_1. Then $\{s_n\}$ is not summed by A but is summed by almost all of its submethods. The $(C, 1)$ method is such that any sequence $\{s_n\}$ which it does not sum is also not summed by almost all of its submethods.

The set of submethods which sums a given sequence is homogeneous so that it must have measure 0 or 1.

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ENTIRE SOLUTIONS OF THE FUNCTIONAL EQUATION $f(f(z)) = g(z)$

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In this note it is proved that: *the functional equation*

$$f(f(z)) = g(z), \tag{1}$$

where $g(z)$ is an entire function of finite order, which is not a polynomial, and which takes on a certain value ρ only a finite number of times, does not have a solution $f(z)$ which is an entire function.

The problem solved here arose from the special case $g(z) = e^z - 1$, which was brought to the author's attention by S. Chowla, who, together with A. J. Kempner and T. Rivlin, had proved that even though a formal power series expansion

$$\sum_{n=1}^{\infty} a_n z^n$$

exists for the solution of the functional equation (1) the function cannot be entire. Further investigations of the author seem to indicate that the radius of convergence of this series is zero.

Other aspects of the subject here discussed are considered in recent articles by Hadamard **(1)**, Isaacs **(2)**, and Kneser **(3)**. These authors also give references to earlier work done in this field.

We begin our proof with an application of a theorem of Polya **(4)**. This theorem states that: if $r(z)$ and $s(z)$ are entire functions, and if $r(s(z))$ is of finite order, then either $s(z)$ is a polynomial, and $r(z)$ is of finite order, or $s(z)$ is not a polynomial but of finite order, and $r(z)$ is of order zero. Applying this theorem to $f(f(z))$, which is to be of finite order, we can conclude that $f(z)$ is of order zero. Also, since $f(f(z))$ is assumed not to be a polynomial, $f(z)$ cannot be a polynomial.

Now let $\{f_m\}$ be the set of all values of z for which $f(z) = \rho$. Here the possibility that this set is empty is included. Denote by $z_k^{(m)}$ all those values of z for which $f(z) = f_m$. Then

$$f(f(z_k^{(m)})) = g(z_k^{(m)}) = f(f_m) = \rho.$$

Since the number of values of z for which $g(z) = \rho$ is finite, and since, in view of the above relation, all $z_k^{(m)}$ have this property, there can only be a finite number of $z_k^{(m)}$. Hence $f(z)$ takes on each value f_m only a finite number of times. According to Picard's theorem there exists at most one value which an entire

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function, not a polynomial, takes on only a finite number of times. It follows either that all f_m are equal to one value, say f^*, or that no f_m, with $f(f_m) = \rho$, exists.

Thus in the first case the only value of z for which $f(z) = \rho$ is f^*, and we can write

$$f(z) = \rho + (z - f^*)^n h(z),$$

where $h(z)$ does not vanish and is an entire function of order zero. It follows from Hadamard's factorization theorem (5, p. 250) that $h(z)$ is a constant. The function $h(z)$ then is a polynomial which we know to be impossible.

In the second case we have immediately, by an analogous argument, that $f(z) = \rho + c$, and thus again a contradiction. Since in the case $g(z) = e^z - 1$ the value -1 is never taken on by $g(z)$ the result mentioned in the beginning of this note is seen to be a special case of our theorem.

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ON AN INVERSION FORMULA FOR THE LAPLACE TRANSFORMATION, II

P. G. ROONEY

In an earlier paper (3) we discussed at some length a certain inversion operator for the Laplace transformation. If

$$f(s) = \int_0^\infty e^{-st}\phi(t)\,dt = \mathscr{L}(\phi(t); s) \qquad \text{I}$$

then the inversion operator is given by

$$'L_{k,t}[f(s)] = \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\int_0^\infty x^{\frac{1}{2}\nu}J_\nu(2kx^{\frac{1}{2}})\,f\left(\frac{k(x+1)}{t}\right)dx, \qquad \text{II}$$

and we showed that if $\nu > -1$, then under certain conditions

$$\lim_{k\to\infty} {}'L_{k,t}[f(s)] = \phi(t).$$

It is our purpose here to discuss the behaviour of the operator for $\nu \leqslant -1$.

It is clear that if $\nu \leqslant -1$ and ν is not an integer, the operator will not exist. For, by (2, §7.2.1, (2)),

$$x^{\frac{1}{2}\nu}J_\nu(2kx^{\frac{1}{2}}) = (kx)^\nu(1 + o(1))/\Gamma(\nu+1), \qquad x \to 0+,$$

if ν is not a negative integer. However, if ν is a negative integer, $\nu = -n$, a different situation appears, for, by (2, §7.2.4, (24)), $J_{-n}(z) = (-1)^nJ_n(z)$, and hence

$$x^{-\frac{1}{2}n}J_{-n}(2kx^{\frac{1}{2}}) = (-1)^nk^n(1 + o(1))/n! \qquad x \to 0+.$$

Thus there is some prospect of the operator existing in this case.

It will transpire that the operator will exist if $\nu = -n$, under certain hypotheses on $\phi(t)$, and that a suitable modification of the operator will invert the transformation. The theory is contained in the following two theorems.

We make use of the notation

$$\int_0^{+\infty}\phi(u)\,du$$

to denote the 'Improper' Lebesgue integral. That is,

$$\int_0^{+\infty}\phi(u)\,du = \lim_{R\to\infty}\int_0^R\phi(u)\,du.$$

Also we define,

$$'\overrightarrow{L}_{k,t}[f(s)] = \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\int_0^{+\infty}x^{\frac{1}{2}\nu}J_\nu(2kx^{\frac{1}{2}})\,f\left(\frac{k(x+1)}{t}\right)dx. \qquad \text{III}$$

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THEOREM 1. *If* $e^{-\gamma t}\phi(t) \in L(0, \infty)$, $\gamma > 0$, *then*

$$^{-n}\overrightarrow{L}_{k,t}[f(s)]$$

exists for each $t > 0$ *and all* $k > \gamma t$, $(n = 1, 2, \ldots)$, *and*

$$^{-n}\overrightarrow{L}_{k,t}[f(s)] = (-t)^{-n}\,{}^{0}L_{k,t}[f^{(n)}(s)] - \frac{(-1)^{n-1}k^{n+\frac{1}{2}}e^{2k}}{\pi^{\frac{1}{2}}}\sum_{r=1}^{n}\frac{(kt)^{-r}}{(n-r)!}f^{(r-1)}\left(\frac{k}{t}\right).$$

Proof. By (1; ch. 3, §2), $f^{(n)}(s) = L((-t)^n\phi(t); s)$, $s > \gamma$, and thus, by (3, Theorem 2.1), ${}^{0}L_{k,t}[f^{(n)}(s)]$ exists for $k > \gamma t$ $(n = 1, 2, \ldots)$.

Let $k > \gamma t$. Then, since

$$\frac{d}{dz}J_0(z) = -J_1(z) = J_{-1}(z),$$

and since $J_0(z)$ and $f(z) \to 0$ as $z \to \infty$, we obtain on integrating by parts,

$$\begin{aligned}
{}^{0}L_{k,t}[f'(s)] &= \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\int_0^{\infty} J_0(2kx^{\frac{1}{2}})\,f'\left(\frac{k(x+1)}{t}\right)dx \\
&= \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\left\{\frac{t}{k}\,J_0(2kx^{\frac{1}{2}})\,f\left(\frac{k(x+1)}{t}\right)\Big|_0^{\infty} - t\int_0^{+\infty} x^{-\frac{1}{2}}J_{-1}(2kx^{\frac{1}{2}})\,f\left(\frac{k(x+1)}{t}\right)dx\right\} \\
&= -\frac{k^{\frac{1}{2}}e^{2k}}{\pi^{\frac{1}{2}}}f\left(\frac{k}{t}\right) - t\cdot{}^{-1}\overrightarrow{L}_{k,t}[f(s)],
\end{aligned}$$

which is the stated result for $n = 1$.

We now proceed by induction. Assuming the result true for n, we have, since $f'(s) = \mathscr{L}(-t\,\phi(t); s)$ for $s > \gamma$, that

$$^{-n}\overrightarrow{L}_{k,t}[f'(s)]$$

exists for $k > \gamma t$ and equals

$$(-t)^{-n}\,{}^{0}L_{k,t}[f^{(n+1)}(s)] - \frac{(-1)^{n-1}k^{n+\frac{1}{2}}}{\pi^{\frac{1}{2}}}\sum_{r=1}^{n}\frac{(kt)^{-r}}{(n-r)!}f^{(r)}\left(\frac{k}{t}\right).$$

Then, for $k > \gamma t$, since, by (2, §7.2.8, (51)),

$$\frac{d}{dz}z^{-n}J_{-n}(z) = z^{-n}J_{-(n+1)}(z)$$

and $J_{-n}(z)$ and $f(z) \to 0$ as $z \to \infty$, we have, on integration by parts, that

$$\begin{aligned}
^{-n}\overrightarrow{L}_{k,t}[f'(s)] &= \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\int_0^{+\infty} x^{-\frac{1}{2}n}J_{-n}(2kx^{\frac{1}{2}})\,f'\left(\frac{k(x+1)}{t}\right)dx \\
&= \frac{k^{3/2}e^{2k}}{t\pi^{\frac{1}{2}}}\left\{\frac{t}{k}\,x^{-\frac{1}{2}n}J_{-n}(2kx^{\frac{1}{2}})\,f\left(\frac{k(x+1)}{t}\right)\Big|_0^{\infty} - t\int_0^{+\infty} x^{-\frac{1}{2}(n+1)}J_{-(n+1)}(2kx^{\frac{1}{2}})\right. \\
&\qquad \left. f\left(\frac{k(x+1)}{t}\right)dx\right\} \\
&= -\frac{(-1)^n k^{n+\frac{1}{2}}e^{2k}}{\pi^{\frac{1}{2}}n!} - t\cdot{}^{-(n+1)}\overrightarrow{L}_{k,t}[f(s)].
\end{aligned}$$

Hence,

$$
\begin{aligned}
{}^{-(n+1)}\overrightarrow{L}_{k,t}[f(s)] &= -t^{-1}\cdot {}^{-n}\overrightarrow{L}_{k,t}[f'(s)] - \frac{(-1)^n k^{n+\frac{1}{2}} e^{2k}}{t\pi^{\frac{1}{2}} n!} f\left(\frac{k}{t}\right) \\
&= (-t)^{-(n+1)}\, {}^{0}L_{k,t}[f^{(n+1)}(s)] + \frac{(-1)^{n-1} k^{n+\frac{1}{2}} e^{2k}}{t\pi^{\frac{1}{2}}} \sum_{r=1}^{n} \frac{(kt)^{-r}}{(n-r)!} f^{(r)}\left(\frac{k}{t}\right) \\
&\qquad - \frac{(-1)^n k^{n+\frac{1}{2}} e^{2k}}{t\pi^{\frac{1}{2}} n!} f\left(\frac{k}{t}\right) \\
&= (-t)^{-(n+1)}\, {}^{0}L_{k,t}[f^{(n+1)}(s)] - \frac{(-1)^n k^{n+3/2} e^{2k}}{\pi^{\frac{1}{2}}} \sum_{r=1}^{n+1} \frac{(kt)^{-r}}{(n+1-r)!} f^{(r-1)}\left(\frac{k}{t}\right).
\end{aligned}
$$

Hence the formula is true for all n.

COROLLARY. *If $e^{-\gamma t}\phi(t) \in L(0, \infty)$, $\gamma > 0$, and if $t^{-\frac{1}{2}}\phi(t) \in L(0, \delta)$, then ${}^{-1}L_{k,t}[f(s)]$ exists and*

$$
{}^{-1}L_{k,t}[f(s)] = -t^{-1}\, {}^{0}L_{k,t}[f'(s)] - \frac{k^{\frac{1}{2}} e^{2k}}{\pi^{\frac{1}{2}} t} f\left(\frac{k}{t}\right).
$$

If $e^{-\gamma t}\phi(t) \in L(0, \infty)$, $\gamma > 0$, then ${}^{-n}L_{k,t}[f(s)]$ exists for $n = 2, 3, \ldots,$ and

$$
{}^{-n}L_{k,t}[f(s)] = (-t)^{-n}\, {}^{0}L_{k,t}[f^{(n)}(s)] - \frac{(-1)^{n-1} k^{n+\frac{1}{2}} e^{2k}}{\pi^{\frac{1}{2}}} \sum_{r=1}^{n} \frac{(kt)^{-r}}{(n-r)!} f^{(r-1)}\left(\frac{k}{t}\right).
$$

Proof. The existence of ${}^{-n}L_{k,t}[f(s)]$ under the various hypotheses follows exactly as in (3, Theorem 2.1). The stated relations now follow from Theorem 1, since

$$
{}^{-n}\overrightarrow{L}_{k,t}[f(s)] = {}^{-n}L_{k,t}[f(s)]
$$

when both exist.

THEOREM 2. *If $e^{-\gamma t}\phi(t) \in L(0, \infty)$, $\gamma > 0$, then at each point $t > 0$ of the Lebesgue set of ϕ,*

$$
\lim_{k\to\infty} \left\{ {}^{-n}\overrightarrow{L}_{k,t}[f(s)] + \frac{(-1)^{n-1} k^{n+\frac{1}{2}} e^{2k}}{\pi^{\frac{1}{2}}} \sum_{r=1}^{n} \frac{(kt)^{-r}}{(n-r)!} f^{(r-1)}\left(\frac{k}{t}\right) \right\} = \phi(t).
$$

Proof. This now follows from Theorem 1, and Theorem 3.1 of (3).

COROLLARY. *If $e^{-\gamma t}\phi(t) \in L(0, \infty)$, $\gamma > 0$, and if $t^{-\frac{1}{2}}\phi(t) \in L(0, \delta)$, for some $\delta > 0$, then at each point $t > 0$ of the Lebesgue set of ϕ,*

$$
\lim_{k\to\infty} \left\{ {}^{-1}L_{k,t}[f(s)] + \frac{k^{3/2} e^{2k}}{\pi^{\frac{1}{2}} tk} f\left(\frac{k}{t}\right) \right\} = \phi(t).
$$

If $e^{-\gamma t}\phi(t) \in L(0, \infty)$, $\gamma > 0$, then at each point $t > 0$ of the Lebesgue set of ϕ,

$$
\lim_{k\to\infty} \left\{ {}^{-n}L_{k,t}[f(s)] + \frac{(-1)^{n-1} k^{n+\frac{1}{2}} e^{2k}}{\pi^{\frac{1}{2}}} \sum_{r=1}^{n} \frac{(kt)^{-r}}{(n-r)!} f^{(r-1)}\left(\frac{k}{t}\right) \right\} = \phi(t)
$$

for $n = 2, 3, \ldots$.

Proof. This now follows from the corollary to Theorem 1, and Theorem 3.1 of **(3)**.

References

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A REGULAR SINGULAR FUNCTIONAL

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1. Introduction. In a joint paper with Leighton **(2)**, the author considered quadratic functionals of the type

$$\int_a^b [r(x)\, y'^2 + p(x)\, y^2]\, dx \qquad (0 < a < b) \tag{1.1}$$

in which $x = 0$ is a singular point of the functional which is otherwise regular on $[0, b]$. The hypothesis on a regular functional includes the assumption that r is continuous and positive on a closed interval $[0, b]$. This assures the existence of extremals for (1.1). As a consequence, the Riccati equation

$$z' - \frac{z^2}{r(x)} + p(x) = 0 \tag{1.2}$$

has continuous solutions, at least locally. If the function r vanishes on an interval, the equation (1.2), and presumably its solutions, is annihilated. However the functional (1.1) is much less affected. This fact has suggested to the author a method of extending the meaning of (1.2). Such is the subject of this article.

In this paper we consider the functional

$$J(y)\Big|_a^b = \int_a^b [r(x)\, y'^2 + 2q(x)\, yy' + p(x)\, y^2]\, dx, \tag{1.3}$$

in order to generalize the Riccati equation

$$z' - \frac{(z + q(x))^2}{r(x)} + p(x) = 0. \tag{1.4}$$

The integral (1.3) is a Lebesgue integral. The functions r, p, and q are measurable functions which are defined on $(-\infty, \infty)$. The functions r and q are bounded on each bounded subinterval of $(-\infty, \infty)$ and p is integrable Lebesgue on each bounded interval. We come to a definition.

A function y is said to be *F_t-admissible* on $[a, b]$ if

(1) y is absolutely continuous on $[a, b]$ and y'^2 is integrable Lebesgue on $[a, b]$;

(2) $y(a) = t$.

We denote by $F_t[a, b]$ the class of all functions y which are F_t-admissible on

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$[a, b]$. We shall be principally concerned with the cases in which $t = 0$ or $t = 1$. By the assumptions on r, p, and q,

$$J(y)\Big|_a^b$$

exists and converges absolutely for every y which is F_t-admissible on $[a, b]$ for some t.

In order to outline the method which we employ, we recall the results for the regular functional and then indicate the points of departure from these. Suppose that r, p, and q are continuous on $(-\infty, \infty)$ and that $r(x) > 0$ there. Then every solution of the Euler equation of (1.2)

$$(r(x)\,y' + q(x)\,y)' - (q(x)\,y' + p(x)\,y) = 0 \tag{1.5}$$

has a continuous derivative on $(-\infty, \infty)$. If u is the unique solution of (1.5) such that $u(b) = 0$, $u'(b) = -1$ and if $u(x) \neq 0$ on (a, b) then the Riccati equation (1.4) has a solution

$$-r(x)\frac{u'(x)}{u(x)} - q(x) \tag{1.6}$$

which is continuous and which has a continuous derivative on (a, b). Further, we have by a well-known formula **(4**, p. 260**)** that for $a < t < b$,

$$J(y)\Big|_t^b = \int_t^b r(x)\left(y'(x) - \frac{u'(x)}{u(x)}\,y(x)\right)^2 dx - y^2(t)\left(r(t)\frac{u'(t)}{u(t)} + q(t)\right). \tag{1.7}$$

As a consequence, if

$$L(t, b) = \min_y J(y)\Big|_t^b, \tag{1.8}$$

where the minimum is taken over all y in $F_1[t, b]$, it follows by (1.7) that $L(t, b)$ exists for every $t \in (a, b)$ and that

$$L(t, b) = -r(t)\frac{u'(t)}{u(t)} - q(t).$$

Moreover, the minimum is attained by the extremal

$$y(x) = \frac{u(x)}{u(t)}.$$

However, once the restriction $r(x) \neq 0$ is removed, the Euler equation (1.5) ceases to exist and the minimum (1.8) will not, in general, be attained. Nevertheless if we define $L(t, b)$ by the relation

$$L(t, b) = \text{g.l.b.}_y\, J(y)\Big|_t^b$$

among all $y \in F_1[t, b]$ we shall see that the functions $L(t, b)$ behave in a manner

which resembles the behavior of solutions of a Riccati equation. In the last section we discuss the details of the case when $r(x) = 0$ for x on $[a, b]$.

2. The conjugate points. In this section we develop a condition which ensures that the function $L(x, b)$ exists on (a, b).

Let $x = a$ be a point of $(-\infty, +\infty)$. If there exists a number b, $b > a$, such that

$$J(y)\Big|_a^b > 0 \tag{2.1}$$

for all y in $F_0[a, b]$, we define $c(a)$ to be the least upper bound of all such b. If no such b exists, we define $c(a)$ to be a. The point $x = c(a)$, which may be $+\infty$, is termed the first *conjugate point* of $x = a$. We remark that this definition is consistent with the definition of the conjugate point for regular functionals **(3**, p. 8**)**. It will be noted that $c(x)$ is a nondecreasing and right continuous function of x. It is not, in general, continuous as trivial examples will show.

THEOREM 2.1. *If* $c(a) \geqslant b$, *then*

$$J(y)\Big|_a^b \geqslant 0$$

for every y *in* $F_0[a, b]$.

This theorem is trivial except in the case when $c(a) = b$. In this case it is disposed of in a manner similar to the proof of **(4**, Theorem 5.2**)**.

We now proceed to consider the condition under which it is possible to define the functions L. We recall from the introduction that $L(x, b)$ is defined to be the number

$$\text{g.l.b.}_y\, J(y)\Big|_x^b \tag{2.2}$$

where the greatest lower bound is taken over all $y \in F_1[x, b]$. If $L(x, b)$ exists on an interval (a, b), we shall call it the *Riccati function* associated with J and the point $x = b$.

THEOREM 2.2. *In order that* $L(x, b)$ *be finite on* (a, b) *it is necessary that* $c(a) \geqslant b$.

The proof is by contradiction. Suppose that $c(a) < b$. Then since c is right continuous, there exists $x_0 > a$ such that $c(x_0) < b$. As a consequence there exists z in $F_0[x_0, b]$ such that

$$J(z)\Big|_{x_0}^b < 0. \tag{2.3}$$

Let $y \in F_1[x_0, b]$. Then for every t, the function $w = y + tz \in F_1[x_0, b]$ and

$$J(w)\Big|_{x_0}^b = J(y)\Big|_{x_0}^b + 2tJ(y, z)\Big|_{x_0}^b + t^2 J(z)\Big|_{x_0}^b \tag{2.4}$$

where

$$J(y, z)\Big|_{x_0}^{b} = \int_{x_0}^{b} [r(x)\, y'z' + q(x)(yz)' + p(x)\, yz]\, dx.$$

Now because of (2.3) it follows that

$$\lim J(w)\Big|_{x_0}^{b} = -\infty$$

and thus $L(x_0, b)$ is not finite.

We now establish the converse of Theorem 2.2.

THEOREM 2.3. *If $c(a) > b$, then $L(x, b)$ is finite for every x on (a, b).*

Let $a < x_0 < x_1 < b$ and let y be an arbitrary function in $F_1[x_1, b]$. Consider any function z which coincides with y on $[x_1, b]$ and which is in $F_0[x_0, b]$. We have then that

$$J(z)\Big|_{x_0}^{b} > 0.$$

Therefore

$$J(y)\Big|_{x_1}^{b} > -J(z)\Big|_{x_0}^{x_1} \tag{2.5}$$

for every y in $F_1[x_1, b]$. Since the right hand side of (2.5) may be independent of y it follows that $L(x_1, b)$ is finite for the arbitrary value x_1 in (a, b) and the theorem is proved.

It follows from the above theorem that, for $a < b_1 < b$ and $a < x < b_1$, $L(x, b_1)$ exists and

$$L(x, b) \leqslant L(x, b_1) \qquad (a < x < b_1). \tag{2.6}$$

3. The Riccati functions. In this section we develop a number of properties of the Riccati functions. As we shall see the function $L(x, b)$ is not, in general, continuous. However, we have the following theorem.

THEOREM 3.1. *If $c(a) > b$ then $L(x, b)$ is right continuous everywhere on (a, b).*

By the previous theorem, $L(x, b)$ is finite for each x in (a, b). Let c be in (a, b) and y be in $F_1[x, b]$. Then y_c defined as follows is a member of $F_1[c, b]$:

$$y_c(t) = \begin{cases} 1, & c < t < x, \\ y(x), & x < t < b. \end{cases}$$

Thus

$$J(y_c)\Big|_{c}^{b} = J(1)\Big|_{c}^{x} + J(y)\Big|_{x}^{b},$$

from which it follows that

$$L(c, b) \leqslant J(1)\Big|_c^x + J(y)\Big|_x^b$$

for every y in $F_1[x, b]$. Therefore

$$L(c, b) \leqslant J(1)\Big|_c^x + L(x, b) \qquad (a < x < b), \tag{3.1}$$

and

$$\liminf_{x=c^+} L(x, b) \geqslant L(c, b). \tag{3.2}$$

Now let y_c be in $F_1[c, b]$. Then

$$y(t) = \frac{y_c(t)}{y_c(x)} \qquad (x \leqslant t \leqslant b),$$

is F_1-admissible on $[x, b]$ if x is sufficiently close to $x = c$ and $x > c$. Now let $\epsilon > 0$ and let y_ϵ in $F_1[c, b]$ be chosen such that

$$J(y_\epsilon)\Big|_c^b < L(c, b) + \epsilon. \tag{3.3}$$

Now

$$J(y)\Big|_x^b = \frac{J(y_c)\Big|_x^b}{y_c^2(x)}$$

and consequently

$$L(x, b) \leqslant \frac{J(y_c)\Big|_x^b}{y_c^2(x)} \tag{3.4}$$

for every y_c in $F_1[c, b]$. It follows then that

$$\limsup_{x=c^+} L(x, b) \leqslant J(y_\epsilon)\Big|_c^b < L(c, b) + \epsilon. \tag{3.5}$$

Since $\epsilon > 0$ is arbitrary,

$$\limsup_{x=c^+} L(x, b) \leqslant L(c, b). \tag{3.6}$$

A comparison of the inequality (3.2) with (3.6) yields the theorem.

We now wish to obtain an extension of equation (1.7). Before stating the next theorem, we note that

$$\text{g.l.b.}_y\, J(y)\Big|_c^b = y^2(c)\, L(c, b) \tag{3.7}$$

where the greatest lower bound is taken over all y in $F_t[c, b]$ for a fixed t.

THEOREM 3.2. *If $c(a) > b$ and if*

$$J(y)\Big|_x^b = Q(y, b)\Big|_x^b + y^2(x)\, L(x, b) \tag{3.8}$$

for every y in $F_c[a, b]$ then

$$Q(y, b)\Big|_x^b$$

is for each x on (a, b), a quadratic functional which is defined for every y in $F_c[x, b]$. For each y in $F_c[a, b]$, it is a positive nonincreasing function of x.

Let y be in $F_c[a, b]$ for some c. We must show that if $a < s < t < b$, then

$$Q(y, b)\Big|_s^t = Q(y, b)\Big|_s^b - Q(y, b)\Big|_t^b > 0. \tag{3.9}$$

Now

$$Q(y, b)\Big|_s^t = \left[J(y)\Big|_s^b - y^2(s)\,L(s, b)\right] - \left[J(y)\Big|_t^b - y^2(t)\,L(t, b)\right] \tag{3.10}$$

or

$$Q(y, b)\Big|_s^t = J(y)\Big|_s^t + y^2(x)\,L(x, b)\Big|_s^t. \tag{3.11}$$

From equation (3.11), it follows that this depends only upon the values of $y(x)$ on the interval $[s, t]$. Now let $z(x)$ be any function of the class $F_c[t, b]$ where $c = y(t)$. We have by the remark above that if

$$\bar{y}(x) = \begin{cases} y(x), & s < x \leqslant t, \\ z(x), & t \leqslant x < b. \end{cases}$$

then

$$\begin{aligned} Q(y, b)\Big|_s^t &= Q(\bar{y}, b)\Big|_s^t \\ &= J(\bar{y})\Big|_s^b - \bar{y}^2(s)\,L(s, b) - J(z)\Big|_t^b + z^2(t)\,L(t, b). \end{aligned} \tag{3.12}$$

Thus by (3.7),

$$Q(y, b)\Big|_s^t > -J(z)\Big|_t^b + z^2(t)\,L(t, b)$$

for every z in $F_c[t, b]$ where $c = y(t)$. On referring again to (3.7) it follows that

$$Q(y, b)\Big|_s^t > 0.$$

THEOREM 3.3. *If $c(a) \geqslant b$, then $L(x, b)$ is of bounded variation on every subinterval of (a, b).*

Letting $y(x) = 1$ and $s = x$ in equation (3.11) we have

$$L(x, b) = -Q(1, b)\Big|_x^t + J(1)\Big|_x^t + L(t, b) \qquad (s < t < b)$$

and since the middle term is absolutely continuous on $[x, t]$ the theorem follows.

THEOREM 3.4. *If* $c(a) > b$ *then*

$$r(x)[L'(x, b) + p(x)] - [L(x, b) + q(x)]^2 \geqslant 0 \tag{3.13}$$

for almost all x *on* (a, b).

A trivial integration by parts provides a proof of the following lemma:

LEMMA 3.1. *If* $f(x)$ *is integrable and* $g(x)$ *is in* C' *on* $[a, b]$ *and if* c *is a point at which* $f(c)$ *is the derivative of*

$$\int_a^x f(t)\, dt$$

then c *is also a point at which* $f(c)\, g(c)$ *is the derivative of*

$$\int_a^x f(t)\, g(t)\, dt.$$

We continue with a proof of the theorem. Let E be a measurable subset of (a, b) such that

$$m(E) = b - a$$

and such that for every c in E, $L'(c, b)$ is finite and the derivatives of the functions

$$\int_x^b r(t)\, dt, \quad \int_x^b q(t)\, dt, \quad \int_x^b p(t)\, dt \tag{3.14}$$

exist and equal $-r(c)$, $-q(c)$ and $-p(c)$ respectively. (Henceforth, if A is a measurable subset of $(-\infty, \infty)$, we will denote its one dimensional Lebesgue measure by $m(A)$.) Now by (3.8) we have for $y = m(x - t) + n$ that

$$Q(y, b)\Big|_x^c = J(y)\Big|_x^c - y^2(t)\, L(t, b)\Big|_x^c \qquad (a < x < c < b) \tag{3.15}$$

and

$$Q(y, b)\Big|_x^b$$

is nonincreasing on (a, b). Further, since $m(x - c) + n$ is of class C^1 on $[a, b]$ we have that for $x = c$ in E, there is a derivative

$$\frac{d}{dx}\, Q(y, b)\Big|_x^b \leqslant 0. \tag{3.16}$$

An application of the Lemma to (3.16) gives the result

$$r(c)m^2 + 2(q(c) + L(c, b))\, mn + (p(c) + L'(c, b))\, n^2 \geqslant 0$$

for every m and n and every c in E.

But from the theory of quadratic forms it follows that

$$r(c) \geqslant 0, \quad p(c) + L'(c, b) \geqslant 0 \tag{3.17}$$

and

$$r(c)[L'(c, b) + p(c)] - [L(c, b) + q(c)]^2 \geqslant 0, \tag{3.18}$$

for all c in E. The theorem is proved.

The relation (3.17) gives at once the following theorem:

THEOREM 3.5. *In order that*

$$J(y)\Big|_a^b \geqslant 0$$

for every y in $F_0[a, b]$ it is necessary that the set of x of $[a, b]$ for which $r(x) < 0$ be of Lebesgue measure zero.

If $r(x)$ is zero on a set T of positive measure then except for a subset of T of measure zero the inequality (3.18) implies that

$$L(x, b) = -q(x).$$

We thus have the theorem:

THEOREM 3.6. *If*

$$R[x, z] = \begin{cases} z' - \dfrac{(z + q(x))^2}{r(x)} + p(x), & r(x) \neq 0, \\ -(z + q(x))^2, & r(x) = 0, \end{cases} \tag{3.19}$$

then

$$R[x, L(x, b)] \geqslant 0 \tag{3.20}$$

almost everywhere on (a, b).

One may show by example that the inequality in relation (3.20) cannot be removed without further assumptions. We have, however, the following result.

THEOREM 3.7. *If $r(x)$ is continuous on $[a, b]$ and if $c(a) \geqslant b$ then $L(x, b)$ satisfies the equation*

$$R[x, z] = 0 \tag{3.21}$$

almost everywhere in (a, b).

Before attending to the proof of Theorem 3.7 we consider some extensions of the theory of regular functionals.

If $r(x)$ is continuous and positive on $[s, t]$ then $1/r(x)$ is continuous on $[s, t]$ and

$$\int_s^t \frac{dx}{r(x)} \neq \infty. \tag{3.22}$$

It follows that the system of equations

$$u' = a_{11}u + a_{12}v, \qquad v' = a_{21}u + a_{22}v, \tag{3.23}$$

where

$$a_{11} = -q/r, \qquad a_{12} = 1/r, \qquad a_{21} = p - q^2/r, \qquad a_{22} = q/r \tag{3.24}$$

has a unique solution (u, v), where u and v are absolutely continuous functions on $[s, t]$ which almost everywhere satisfy equations (3.23) and which at c in $[s, t]$ satisfy the relations

$$u(c) = u_0, \qquad v(c) = v_0,$$

where u_0 and v_0 are arbitrary real numbers. This follows **(5**, pp. 44–45**)** from the fact that the a_{ij} are integrable Lebesgue on $[s, t]$. By the null solution of (3.23) is meant the solution for which u and v vanish identically on an interval. It follows from the existence theorem cited that if u and v vanish at one point of (s, t) then (u, v) is the null solution on $[s, t]$. From (3.23) and (3.24) we have that for almost all x in [s, t].

$$v(x) = r(x)\,u'(x) + q(x)\,u(x) \tag{3.25}$$

and

$$v'(x) = q(x)\,u'(x) + p(x)\,u(x). \tag{3.26}$$

Furthermore, in a manner which runs along classical lines, one may show that the first conjugate point $c(a)$ of the point $x = a$ is the first zero beyond the point a of u of a nonnull solution (u, v) of (3.23) for which $u(a) = 0$. Hence if $c(a) > b$ the function

$$-\frac{v(x)}{u(x)} \tag{3.27}$$

is absolutely continuous on every closed subinterval of (a, b) and satisfies the Riccati equation

$$z' - \frac{(z + q(x))^2}{r(x)} + p(x) = 0 \tag{3.28}$$

for almost all x on (a, b). We have moreover that

$$J(y)\Big|_z^b = \int_z^b r(t)\left[y'(t) - \frac{u'(t)}{u(t)}\,y(t)\right]^2 dt - \frac{v(x)}{u(x)}\,y^2(x) \tag{3.29}$$

for every y in $F_1[a, b]$. The proof of this formula is similar to that of the analogous formula in **(2**, p. 103, Th. 6.2**)**. A consequence of (3.29) is the following well-known result.

THEOREM 3.8. *If $c(a) > b$ and if (u, v) is any nonnull solution of the system* (3.23) *such that $u(b) = 0$ then*

$$L(x, b) = -\frac{v(x)}{u(x)} \qquad (a < x < b). \tag{3.30}$$

We return to give a proof of Theorem 3.7. Let r_n denote the function such that

$$r_n(x) = r(x) + \frac{1}{n} \qquad (a < x < b), \tag{3.31}$$

$$J_n(y)\Big|_x^b = \int_x^b [r_n(t)\, y'^2 + 2q(t)\, yy' + p(t)\, y^2]\, dt. \tag{3.32}$$

Let $L_n(x, b)$ denote the Riccati functions of J_n. We then have that

$$J_n(y)\Big|_x^b > J_{n+1}(y)\Big|_x^b > J(y)\Big|_x^b \tag{3.33}$$

from which it follows that

$$L_n(x, b) > L_{n+1}(x, b) > L(x, b) \qquad (a < x < b). \tag{3.34}$$

Thus for each x on (a, b) there is a limit

$$\lim_{n=\infty} L_n(x, b) > L(x, b). \tag{3.35}$$

We assert that

$$\lim_{n=\infty} L_n(x, b) = L(x, b). \tag{3.36}$$

Since for each n, and every y in $F_1[x, b]$,

$$0 < r_n(t)\, y'^2(t) < r_1(t)\, y'^2(t) \qquad (x < t < b),$$

the Lebesgue limit theorem gives the result

$$\lim_{n=\infty} J_n(y)\Big|_x^b = J(y)\Big|_x^b \qquad (a < x < b). \tag{3.37}$$

Now let y be in $F_1[x, b]$ such that

$$J(y)\Big|_x^b < L(x, b) + \epsilon$$

and let m be an integer such that for $n > m$

$$J_n(y)\Big|_x^b < L(x, b) + \epsilon.$$

By (3.37) such an integer m exists. Thus

$$\lim_{n=\infty} L_n(x, b) < \lim_{n=\infty} J_n(y)\Big|_x^b < L(x, b) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary it follows that

$$\lim_{n=\infty} L_n(x, b) < L(x, b),$$

and combining this result with (3.35) gives the assertion (3.36). Now since $1/r_n(x) \in L_1$ on every closed subinterval of (a, b) it follows from the remark (3.27) that $L_n(x, b)$ is a solution of the Riccati equation

$$z' - \frac{(z + q(x))^2}{r_n(x)} + p(x) = 0 \qquad (a < x < b). \tag{3.38}$$

Let T be the subset x on (a, b) for which $r(x) = 0$ and S the subset consisting of x on (a, b) for which $r(x) > 0$. Then

$$S \cup T = (a, b),$$

and since $r(x)$ is continuous on (a, b) S is open while T is closed in (a, b). If $m(T) < 0$ then by the inequality (3.18)

$$L(x, b) = -q(x)$$

for almost all of T, and hence

$$R[x, L(x, b)] = 0 \tag{3.39}$$

on almost all of T. If S is not empty then $m(S) > 0$. In the latter case let I_k be the components of S, let

$$I_k = (a_k, b_k) \qquad (k = 1, 2, \ldots).$$

It suffices to show that (3.39) holds on almost all of each interval I_k. Let k be a fixed but arbitrary integer and let $[s, t]$ be a closed subinterval of I_k. We have then by Theorem 3.8 that $L_n(x, b)$ is absolutely continuous on each closed subinterval of (a, b). Thus by (3.20) we have that

$$L_n(t, b) - L_n(s, b) = \int_s^t \frac{(L_n(x, b) + q(x))^2}{r(x)}\,dx - \int_s^t p(x)\,dx$$

Now since, for every n,

$$\frac{(L_n(x, b) + q(x))^2}{r_n(x)} \leqslant \frac{(|L_1(x, b)| + |q(x)|)^2}{r(x)},$$

the Lebesgue limit theorem may be invoked to infer that

$$L(t, b) - L(s, b) = \int_s^t \frac{(L(s, b) + q(x))^2}{r(x)}\,dx - \int_s^t p(x)\,dx \tag{3.40}$$

for every t and s such that

$$a_k < s < t < b_k.$$

On taking the derivative of (3.40) when $t < x$ we have almost everywhere on I_k that

$$L'(x, b) - \frac{(L(x, b) + q(x))^2}{r(x)} + p(x) = 0.$$

The proof is complete.

We remark that equation (3.40) contains the following result.

THEOREM 3.9. *If $c(a) > b$ then $L(x, b)$ is absolutely continuous on every closed interval in (a, b) where $r(x)$ is continuous and positive.*

As the results of the next section will show, even if $r(x)$ is continuous, the condition that $c(a) > b$ does not imply that $L(x, b)$ is continuous.

Let $r(x)$ be continuous on $(-\infty, \infty)$. Let $I_k = (a_k, b_k)$ be the components of the subset S of (a, b) on which $r(x) > 0$. Let $x = m_k$ be a point of (a_k, b_k) and T the complement of S in (a, b). We have the following theorem.

THEOREM 3.10. *Let $c(a) > b$. If $q(x)$ is continuous on T and if*

$$\int_{a_k}^{m_k} \frac{dx}{r(x)} = \infty, \quad \int_{m_k}^{b_k} \frac{dx}{r(x)} = \infty \qquad (k = 1, 2, \ldots) \tag{3.41}$$

then $L(x, b)$ is continuous on (a, b) and $L(x, b) = -q(x)$ for every x in T.

By Theorem 3.9 it follows at once that $L(x, b)$ is continuous at each point of S. To show that L is continuous at each point $x = c$ of T it is therefore sufficient to prove that

$$L(c, b) = -q(c) \tag{3.42}$$

and

$$\lim_{x=c_n} L(c_n, b) = -q(c) \tag{3.43}$$

for one sequence of numbers such that $c_n < c$ for all $n = 1, 2, \ldots$. This follows from the facts that $L(x, b)$ is right continuous on (a, b) and that it is of bounded variation on every closed subinterval of (a, b).

We assert first that

$$L(c, b) = -q(c)$$

for every point $x = c$ in T. By the hypothesis (3.41) and the formula

$$L(m_k, b) - L(a_k, b) = \int_{a_k}^{m_k} \frac{(L(x, b) + q(x))^2}{r(x)}\, dx - \int_{a_k}^{m_k} p(x)\, dx,$$

it follows that

$$L(a_k, b) = \lim_{x=a_k} L(x, b) = -q(a_k). \tag{3.44}$$

Let $x = c$ be any point of T. If $x = c$ is a right limit point of the sequence a_k the assertion (3.42) follows from (3.44). In the remaining case $x = c$ is isolated on the right from a_k. Then T contains an interval (c, a'), $(0 < a' < b)$. But since $r(x) = 0$ in $[c, a']$, Theorem 3.6 implies that

$$L(x, b) = -q(x)$$

almost everywhere in $[c, a']$, in particular on an everywhere dense subset of $[c, a']$. Then on taking c_n in $[c, a']$ such that

$$L(c_n, b) = -q(c_n) \qquad (n = 1, 2, \ldots),$$

the assertion follows from the continuity of q and the right continuity of L. Now the desired result (3.43) is clear for every point of T which is a left limit point of T. If $x = c$ is isolated on the left from T, then $x = c$ is the right endpoint of an interval of S. Thus $c = b_k$ and by reasoning analogous to that used for the a_k we have

$$\lim_{x=b_k^-} L(x, b) = -q(b_k, b) = L(b_k, b).$$

4. The case $r = 0$. In this section it will be presumed throughout that $r(x)$ vanishes identically on $(-\infty, \infty)$. We will determine the Riccati functions completely and obtain a necessary and sufficient conjugate point condition.

For the functionals under consideration

$$R[x, z] = -(z + q(x))^2. \tag{4.1}$$

Further since $r = 0$ is everywhere continuous, $L(x, b)$, whenever it exists, satisfies the Riccati equation

$$R[x, z] = 0$$

almost everywhere on (a, b). Thus for these x,

$$L(x, b) = -q(x). \tag{4.2}$$

We have then as an immediate consequence of (4.2), Theorem 3.1, and Corollary 3.1 the following result.

THEOREM 4.1. *If $c(a) > b$ then $q(x)$ must agree almost everywhere on (a, b) with a function which is right continuous on (a, b) and which is of bounded variation on every closed subinterval of (a, b).*

With no loss of generality we may always assume that whenever $c(a) > b$ that $q(x)$ is right continuous on (a, b) and is of bounded variation on every closed subinterval of (a, b).

THEOREM 4.2. *In order that $c(a) > b$ it is necessary that the function*

$$q(x) + \int_x^b p(t)\, dt \tag{4.3}$$

be nonincreasing on (a, b).

LEMMA 4.1. *If $x = c$ is a point where $q(x)$ is right continuous then*

$$\lim_{x=c^+} J\left(\frac{x - t}{c - t}\right)\Bigg|_c^t = -q(c) \tag{4.4}$$

and

$$\lim_{x=c^+} J\left(\frac{x - c}{t - c}\right)\Bigg|_c^t = q(c). \tag{4.5}$$

We shall provide a demonstration of equation (4.4).

It is sufficient to show that

$$\lim_{t=c^+} \int_c^t 2q(x) \frac{(x-t)}{(c-t)^2}\,dx = -q(c) \tag{4.6}$$

since

$$\left| \int_c^t p(t) \frac{(x-t)^2}{(c-t)^2}\,dx \right| \leqslant \int_c^t |p(x)|\,dx = O(1)$$

as t tends to c^+. But on an integration by parts we find that

$$\frac{2}{(c-t)^2} \int_c^t q(x)(x-t)\,dx = \frac{-2}{(c-t)^2} \int_c^t \int_c^s q(x)\,dx\,ds.$$

By l'Hospital's theorem one has

$$\lim_{t=c^+} \frac{2}{(c-t)^2} \int_c^t q(x)(x-t)\,dx = \lim_{t=c^+} \frac{-1}{t-c} \int_c^t q(x)\,dx = -q(c),$$

since $q(x)$ is right continuous at $x = c$.

The lemma is proved. We continue with a proof of the theorem.

Let $F(x)$ denote the function (4.3). As previously remarked we may assume that $F(x)$ is right continuous and of bounded variation on every closed subinterval of (a, b). Let $x = s$ and $x = u$ be two points of (a, b) such that $s < u$, and let numbers t and v be chosen such that

$$s < t < u < v. \tag{4.7}$$

We define a function y in $F_0[a, b]$ as follows:

$$y(x) = \begin{cases} 0, & a \leqslant x \leqslant s, \\ \dfrac{x-s}{t-s}, & s \leqslant x \leqslant t, \\ 1, & t \leqslant x \leqslant u, \\ \dfrac{x-v}{u-v}, & u \leqslant x \leqslant v, \\ 0, & v \leqslant x \leqslant b. \end{cases}$$

Since $c(a) > b$, we have

$$J(y)\Big|_a^b > 0,$$

or what is the same thing:

$$J\left(\frac{x-s}{t-s}\right)\Big|_s^t + \int_t^u p(x)\,dx + J\left(\frac{x-v}{u-v}\right)\Big|_u^v > 0 \tag{4.8}$$

for every s, t, u, and v which satisfy the inequalities (4.7). Now let t tend to s^+ and v then to u^+. By Lemma 4.1 the inequality (4.8) becomes

$$q(s) + \int_s^u p(x)\,dx - q(u) \geqslant 0,$$

that is,

$$F(s) - F(u) \geqslant 0 \qquad (a < s < u < b),$$

and the theorem follows.

We now establish the converse of the preceding Theorem.

THEOREM 4.3. *If the function*

$$q(x) + \int_x^b p(t)\, dt \tag{4.9}$$

is nonincreasing on (a, b) *then* $c(a) \geqslant b$.

Again we denote (4.9) by $F(x)$. The theorem will be proved if we show that

$$J(y)\Big|_a^b \geqslant 0$$

for every y in $F_0[a, b]$. On an integration by parts one finds that for every y in $F_0[a, b]$

$$J(y)\Big|_a^b = 2\int_a^b F(x)\, y(x)\, y'(x)\, dx. \tag{4.10}$$

Since $F(x)$ is nonincreasing we may infer from the Second Mean-Value Theorem that there exists c on (a, b) such that

$$\begin{aligned} J(y)\Big|_a^b &= 2F(a)\int_a^c y(x)\, y'(x)\, dx + 2F(b)\int_c^b y(x)\, y'(x)\, dx \\ &= (F(a) - F(b))\, y^2(c) \geqslant 0. \end{aligned}$$

The proof is complete.

We remark in closing that if

$$L(x, b) = -q(x)$$

on a set of positive measure T then there exists a subset T' such that

$$m(T') = m(T)$$

such that

$$L(x, c) = -q(x)$$

for all x in T' and all c such that

$$x < c < b.$$

Hence under these circumstances $L(x, b)$ is independent of b. Further, if $r(x) \equiv 0$, $L(x, b)$ is independent of the function p subject only to the condition that the function

$$q(x) + \int_x^b p(t)\, dt$$

be nonincreasing on (a, b).

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A CONDITION FOR EXISTENCE OF MOMENTS OF INFINITELY DIVISIBLE DISTRIBUTIONS

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Let $F(x)$ be an infinitely divisible distribution and let $\phi(t)$ be its characteristic function. As is well known according to the formula of Lévy and Khintchine, $\phi(t)$ has the following representation:

$$\phi(t) = \exp\left\{i\gamma t + \int_{-\infty}^{\infty}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}\,dG(u)\right\} \tag{1}$$

where γ is a real constant and $G(u)$ is a bounded nondecreasing function. A simple necessary and sufficient condition for the moments of $F(x)$ to exist is contained in the following Theorem.

THEOREM. *A necessary and sufficient condition for the $(2k)$th moment of $F(x)$ to be finite is that the $(2k)$th moment of $G(u)$ be finite where k is any positive integer.*

Proof. To prove sufficiency we assume

$$\int_{-\infty}^{\infty} u^{2k}\,dG(u) < \infty$$

and note that this implies that

$$\alpha(t) \equiv \int_{-\infty}^{\infty}\left(e^{itu} - 1 - \frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}\,dG(u)$$

may be differentiated under the integral sign $2k$ times. Hence using (1) it follows that $d^{2k}\phi(t)/dt^{2k}$ exists and is finite. This implies that

$$\int_{-\infty}^{\infty} x^{2k}\,dF(x) < \infty,$$

(1, p. 90**)**, which proves sufficiency.

Now assuming that

$$\int_{-\infty}^{\infty} x^{2k}\,dF(x) < \infty$$

it follows that $\phi(t)$ has a finite derivative of order $2k$ and thus from (1) we see that $\alpha(t)$ has a finite $(2k)$th derivative. Now

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$$\alpha''(t) = \lim_{h\to 0} \frac{\alpha(t+2h) - 2\alpha(t) + \alpha(t-2h)}{4h^2}$$

so that

$$\begin{aligned}\alpha''(0) &= \lim_{t\to 0} \int_{-\infty}^{\infty} (e^{2itu} - 2 + e^{-2itu}) \frac{1+u^2}{4t^2u^2}\, dG(u) \\ &= \lim_{t\to 0} \int_{-\infty}^{\infty} [(e^{itu} - e^{-itu})/2t]^2 \frac{1+u^2}{u^2}\, dG(u) \\ &= -\lim_{t\to 0} \int_{-\infty}^{\infty} \left(\frac{\sin tu}{t}\right)^2 \frac{1+u^2}{u^2}\, dG(u).\end{aligned}$$

But by Fatou's Theorem

$$\int_{-\infty}^{\infty} (1+u^2)\, dG(u) \leqslant \lim_{t\to 0} \int_{-\infty}^{\infty} \left(\frac{\sin tu}{t}\right)^2 \frac{1+u^2}{u^2}\, dG(u)$$

and hence

$$\int_{-\infty}^{\infty} u^2 dG(u) < \infty.$$

But as in the sufficiency proof this implies that we may differentiate $\alpha(t)$ under the integral sign twice and we see

$$\alpha''(t) = -\int_{-\infty}^{\infty} (1+u^2)\, e^{itu} dG(u) = \beta(t).$$

Now $\beta(t)$ has a finite derivative of order $2k-2$ so that using an argument identical to that of **(1**, p. 90**)** we see that

$$\int_{-\infty}^{\infty} u^{2k} dG(u) < \infty.$$

This proves the Theorem.

It is not difficult to find the relation between the moments of $F(x)$ and those of $G(u)$. In fact if we assume that the $(2k)$th moment of $F(x)$ (or of $G(u)$) is finite it follows that the semi-invariants of $F(x)$ (which are expressible in terms of the moments of $F(x)$)

$$\chi_r = i^{-r}[d^r \log \phi(t)/dt^r]_{t=0}$$

exist for $1 \leqslant r \leqslant 2k$. But

$$\log \phi(t) = i\gamma t + \int_{-\infty}^{\infty} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right) \frac{1+u^2}{u^2}\, dG(u)$$

and differentiating under the integral sign, since

$$\int_{-\infty}^{\infty} u^{2k} dG(u) < \infty,$$

we see that

$$\chi_1 = \gamma + \int_{-\infty}^{\infty} u dG(u),$$

$$\chi_r = \int_{-\infty}^{\infty} (u^{r-2} + u^r)\, dG(u), \qquad 2 < r < 2k.$$

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ON DISCONJUGATE DIFFERENTIAL SYSTEMS

PHILIP HARTMAN and AUREL WINTNER

1. Introduction. Let F, G (and all capital letters to be used below) denote n by n matrices the elements of which are *real-valued* continuous functions on an interval $a \leqslant t \leqslant b$. Correspondingly, by a solution $x = x(t)$ of a differential system

$$(G(t)x')' + F(t)\,x = 0, \tag{1}$$

with $\det G(t) \neq 0$, or of a differential system

$$x'' + G(t)\,x' + F(t)\,x = 0, \tag{2}$$

will be meant a (vector) solution all n components of which are real-valued.

The trivial solution $x(t) \equiv 0$ will not be referred to as a solution. Following the manner of speaking introduced in **(7)** in the scalar case ($n = 1$), let (1) or (2) be called disconjugate on the interval $a \leqslant t \leqslant b$ if there does not exist a solution $x(t)$ satisfying

$$x(t_1) = 0 \text{ and } x(t_2) = 0 \text{ if } a \leqslant t_1 < t_2 \leqslant b. \tag{3}$$

The following considerations deal with various facts concerning disconjugate vector equations, (1) or (2), for an arbitrary n. Sections 3–5 concern an extension of a result **(2**, p. 66**)** from self-adjoint systems (1) to non-self-adjoint systems (1). Sections 6–9 deal with criteria which are sufficient in order that the system (2) be disconjugate; these criteria can be considered analogues of the trivial condition $F \leqslant 0$ in the case that (2) is a scalar equation. Section 10 concerns convexity properties of solutions of (2). In §11, these properties are used to obtain an extension of the result of **(6)** on the half-line $0 \leqslant t < \infty$.

2. Notations. Since every matrix $A = (a_{ik})$ is supposed to be real, its transposed matrix (a_{ki}) will be denoted by A^*. Correspondingly, the symmetric part of A is its Hermitian part, that is, the matrix $\frac{1}{2}(A + A^*)$. The latter will be denoted by A°. Thus $A = A^\circ$ if and only if A is symmetric ($= A^*$).

By $A \leqslant B$ or $B \geqslant A$ will be meant that the quadratic form belonging to $(B - A)^\circ = B^\circ - A^\circ$ is non-negative definite (that is, that $(B - A)^\circ \geqslant 0$, where 0 denotes the zero matrix). Accordingly, $A \leqslant B$ is equivalent to $A^\circ \leqslant B^\circ$. Similarly, $A < B$ or $B > A$ is defined to mean that (the quadratic form belonging to the symmetric matrix) $B^\circ - A^\circ$ is positive definite (that is, $A^\circ < B^\circ$ or $B^\circ > A^\circ$).

3. On a theorem of Morse. In terms of the definition given in connection with (3), the simplest case of a theorem of Morse **(6**, p. 66**)**, a theorem which

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extends Sturm's comparison theorem from the scalar case to the case of vectors, can be formulated as follows:

(I) (Morse) *Suppose that the pair of coefficient matrices P_j, Q_j of the linear differential systems*

$$(P_j(t)x')' + Q_j(t)x = 0, \tag{4_j}$$

where $j = 1, 2$, satisfy the following conditions at every t on $a \leqslant t \leqslant b$:

$$0 < P_1 \leqslant P_2, \quad Q_1 \geqslant Q_2, \tag{5}$$

and

$$P_j = P_j^*, \quad Q_j = Q_j^*. \tag{6_j}$$

Then (4_2) is disconjugate on $a \leqslant t \leqslant b$ whenever (4_1) is disconjugate there.

Here the real matrices P_j, Q_j (and later on $F, G, \ldots$) are supposed to be given as continuous functions on $a \leqslant t \leqslant b$.

It will turn out that the above-quoted consequence of Morse's comparison theorem can be refined so as to *eliminate for* (4_2), but not for (4_1), *the restriction to be self-adjoint*, that is, the restriction (6_2):

(II) *The assertion of* (I) *remains true if, instead of* (5), (6_1) *and* (6_2), *only* (5) *and* (6_1) *are assumed.*

It follows from the refinement (II) of (I) that, without assuming (1) to be self-adjoint, the disconjugate character of (1) can be assured by exhibiting for (1) a *scalar* Sturmian majorant. In fact, it is clear that (II) contains the following

COROLLARY. *Let $p(t)$ and $q(t)$ be scalar continuous functions satisfying*

$$0 < p(t)I \leqslant G(t) \text{ and } q(t)I \geqslant F(t), \tag{7}$$

where I is the unit matrix and $a \leqslant t \leqslant b$. Then (1) *must be disconjugate on $a \leqslant t \leqslant b$ whenever the scalar equation*

$$(p(t)\,x')' + q(t)\,x = 0 \tag{8}$$

is disconjugate there.

Note that (7) implies that $G(t) > 0$. Conversely, if $G(t) > 0$, then (7) can be satisfied by choosing $p(t)$ to be the least eigenvalue of $G^\circ(t)$, and $q(t)$ the greatest eigenvalue of $F^\circ(t)$, for every fixed t.

4. Reduction to the self-adjoint case. Clearly, the extension (II) of (I) can be concluded from (I) and from the following

LEMMA 1. *If the differential system*

$$(G^\circ(t)\,x')' + F^\circ(t)\,x = 0, \tag{9}$$

where $G^\circ(t) > 0$, is disconjugate on $a \leqslant t \leqslant b$, then (1) *is disconjugate there.*

It should be noted that the assumption $\det G \not\equiv 0$ in (1) is implied by the assumption $G^\circ > 0$ in (9); in fact, it is known (cf. **3**) that $\det G \geqslant \det G^\circ$ whenever $G^\circ > 0$.

The proof of Lemma 1 will depend on an adaptation of Morse's proof of (I). Before proving Lemma 1, it will be shown that its converse is false: *If* (1) *is disconjugate on* $a \leqslant t \leqslant b$, *then* (9) *need not be, not even when* $G^\circ > 0$ (*hence* $\det G^\circ > 0$). This is shown by the following example:

Let $n = 2$ and $a = 0$, $b = \pi$, let G (hence G°) be the unit matrix and let

$$F = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \text{ hence } F^\circ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } x = \begin{pmatrix} u \\ v \end{pmatrix}.$$

Then (1) and (9) become

$$u'' = 0,\ v'' + 2u = 0 \text{ and } u'' + v = 0,\ v'' + u = 0,$$

respectively. But the general solution of the first of these two systems is

$$u = c_1 t + c_2, \quad v = -\tfrac{1}{3}c_1 t^3 - c_2 t^2 + c_3 t + c_4$$

which implies that if both u and v vanish at some $t = t_1$ and another $t = t_2$, then u and v, hence the vector x, must vanish for all t. In other words, the present case of (1) is disconjugate on every interval $a \leqslant t \leqslant b$. On the other hand, the present case of (9) fails to be disconjugate on $0 \leqslant t \leqslant \pi$, since $(u, v) = (\sin t, \sin t)$ is a solution satisfying $x(0) = 0 = x(\pi)$.

5. Proof of Lemma 1. Suppose that the assertion of Lemma 1 is false. Then the assumptions of Lemma 1 are satisfied but (1) is not disconjugate on $a \leqslant t \leqslant b$; so that (1) has a solution $x(t) \not\equiv 0$ satisfying (3) for some t_1, t_2. Since the definition of disconjugate behavior on an interval is such as to imply disconjugate behavior on any subinterval of that interval, it can be assumed that $t_1 = a$ and $t_2 = b$, hence

$$x(a) = 0, \quad x(b) = 0, \tag{10}$$

but $x(t) \not\equiv 0$. Scalar multiplication of (1) by $x(t)$, when followed by integration over $a \leqslant t \leqslant b$ and a partial integration of the result, leads to

$$\int_a^b (x' \,.\, Gx' - x \,.\, Fx)\, dt = 0,$$

by (10). This can be written in the form

$$\int_a^b (x' \,.\, G^\circ x' - x \,.\, F^\circ x)\, dt = 0, \tag{11}$$

since $z \,.\, Az = z \,.\, A^\circ z$, by the definition of A° in §2.

On the other hand, since (9) is supposed to be disconjugate on $a \leqslant t \leqslant b$, it is clear that $\lambda_0 > 0$, if λ_0 denotes the least eigenvalue of the boundary value problem (10) of (9). But $\lambda_0 > 0$ means that

$$\min \int_a^b (x' \,.\, G^0 x' - x \,.\, F^0 x)\, dt > 0, \tag{12}$$

where the min refers, for instance, to the class of all continuously differentiable vector functions $x(t)$, $a \leqslant t \leqslant b$, satisfying (10) and

$$\int_a^b x \,.\, x\, dt = 1. \tag{13}$$

Since (12) and (13) contradict (11), the proof of Lemma 1 is complete.

6. A generalization of the Schwarz invariant. The following considerations will deal with (2), rather than (1). The results of the next section can be motivated by the remark that the matrix

$$E = F - \tfrac{1}{2}G' - \tfrac{1}{4}G^2 \tag{14}$$

reduces to Schwarz's invariant (cf. e.g., **5**, p. 121) in the case that (2) is a scalar equation. The connection between (2) and (14) is given by the following

LEMMA 2. *Let $F(t)$ be continuous, and $G(t)$ continuously differentiable, on $a \leqslant t \leqslant b$. Let $R = R(t)$ be a solution of*

$$R' = -\tfrac{1}{2}G(t)\, R \text{ and } \det R \not\equiv 0. \tag{15}$$

Then the substitution

$$x = R(t)y \tag{16}$$

transforms (2) *into*

$$y'' + B(t)y = 0, \quad B = R^{-1}ER \tag{17}$$

where $E = E(t)$ is given by (14).

The existence of matrices $R(t)$ satisfying (15) is clear. In fact, such a matrix results by placing $R = (r_1, \ldots, r_n)$, where the column vectors $r_k = r_k(t)$ are n linearly independent, and otherwise arbitrary, solution vectors of the linear system $r' = -\frac{1}{2}G(t)r$.

Proof of Lemma 2. Since $G(t)$ is assumed to have a continuous derivative, it follows from (15) that $R(t)$ has a continuous second derivation given by

$$R'' = -\tfrac{1}{2}G'R - \tfrac{1}{2}GR' = -\tfrac{1}{2}G'R + (\tfrac{1}{2}G)^2R. \tag{18}$$

A formal substitution of (16) into (2) gives the equation

$$Ry'' + (2R' + GR)y' + (R'' + GR' + FR)y = 0. \tag{19}$$

The coefficient of y' is 0, by virtue of (15). Hence, (19) can be written as $y'' + By = 0$, where

$$B = R^{-1}(R''R^{-1} + GR'R^{-1} + F)R. \tag{20}$$

Since $R'R^{-1} = -\frac{1}{2}G$, by (15), and $R''R^{-1} = -\frac{1}{2}G' + (\frac{1}{2}G)^2$, by (18), it follows that the matrix (20) is $R^{-1}ER$. This gives (17), hence Lemma 2.

Remark. In order to apply comparison theorems to (17), it is necessary to know bounds for the quadratic form $x \cdot Bx$. In general, a knowledge of such bounds depends on a knowledge of R. An exceptional case is the case of the following remark: *If the coefficient $G(t)$ in* (2) *is skew-symmetric for every t, then $R(t)$ in* (17) *can be chosen to be a rotation matrix for every t*; that is, $R^* = R^{-1}$ and $\det R = +1$, so that $B(t)$ is orthogonally equivalent to $E(t)$ for every fixed t. This can be seen as follows:

If $G^* = -G$, then it is readily verified from (15) and $(R^*)' = (R')^*$ that $R^*R' + R^{*\prime}R = 0$; so that $R^*R = \text{const.}$ for every solution $R = R(t)$ of (15). Hence, if the matrix $R(t)$ is determined by the initial condition $R(a) = I$, then $R^*R = \text{const.} = I$, and the assertion follows.

7. Criteria for (2) to be disconjugate. By adapting to matrices a device used repeatedly by Picard (cf., e.g., **4**, p. 8), the following criteria will now be proved:

(III) *In order that* (2) *be disconjugate on $a \leqslant t \leqslant b$, it is sufficient that, for $a \leqslant t \leqslant b$, either*

$$F \leqslant -\tfrac{1}{4}GG^* \tag{21}$$

or

$$F \leqslant G' - \tfrac{1}{4}G^*G, \tag{22}$$

where the existence of a continuous derivative G' is part of the assumption in the second case (but not in the first).

Note that, since $G(t)$ and $G^*(t)$ are not in general commutable, (22) does *not* reduce to (21) even if $G = \text{const.}$ (i.e., $G' = 0$).

Since GG^* is a (symmetric) non-negative definite matrix for every G, condition (21) is relaxed if the $\frac{1}{4}$ is replaced by $\frac{1}{4} - \epsilon < \frac{1}{4}$. But it turns out that any such relaxation of (21) is inadmissible: *In* (III), *the constant $\frac{1}{4}$, occurring in* (21) *and* (22), *is the best possible constant.* This can be seen as follows:

Let $n = 2$ and

$$G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{23}$$

finally $F = \lambda I$, where λ is a constant. Then, since $GG^* = I = G^*G$ and $G' = 0$ in view of (23), it is clear that the condition in which the $\frac{1}{4}$ of (21) or (22) is relaxed to $\frac{1}{4} - \epsilon$ is satisfied if $\lambda \leqslant -\frac{1}{4} + \epsilon$, for example, if

$$F = (-\tfrac{1}{4} + \epsilon)I \qquad (\epsilon > 0). \tag{24}$$

Hence it is sufficient to show that the case (23)–(24) of (2) fails to be disconjugate on the interval $a \leqslant t \leqslant b$, if the latter is suitably chosen (the matrices (23), (24) are independent of t).

First, since (23) implies that $G' = 0$ and $G^2 = I$, it follows from (14) and (24) that $E = \epsilon I$. Hence the differential equation in (17) reduces to $y'' + \epsilon y = 0$

and admits therefore the (vector) solution $y(t) = I \sin(\epsilon^{\frac{1}{2}}t)$. Since the latter vector has more than one zero on $a \leqslant t \leqslant b$ if ϵ is fixed and $b - a$ is large enough, it follows from (16), where $\det R(t) \neq 0$, that (2) has a solution satisfying (10) if the interval $a \leqslant t \leqslant b$ is suitably chosen. This proves the last italicized statement.

Instead of proving (III) directly, the following generalization of (III) will be derived:

(IV) *In order that* (2) *be disconjugate on* $a \leqslant t \leqslant b$, *it is sufficient that there exists a continuously differentiable matrix* $K(t)$ *satisfying*

$$F - K' \leqslant -(\tfrac{1}{2}G - K^\circ)(\tfrac{1}{2}G^* - K^\circ) \tag{25}$$

on $a \leqslant t \leqslant b$.

This criterion implies both assertions of (III). In fact, (25) reduces to (21) if $K = 0$, and to (22) if $K = G$.

8. Proof of (IV). Multiply (2) by x, substitute the value of the scalar product $x \,.\, x''$ from the identity $x \,.\, x'' = (x \,.\, x')' - x'^2$ and, under the assumption (3), integrate the result between t_1 and t_2. This gives the relation

$$\int_{t_1}^{t_2} (x'^2 - x \,.\, Gx' - x \,.\, Fx)\, dt = 0. \tag{25a}$$

If $K = K(t)$ is any matrix function possessing a continuous derivative, then, since

$$(x \,.\, Kx)' = x' \,.\, (K + K^*)x + x \,.\, K'x$$

and $K + K^* = 2K^\circ$, and since

$$\int_{t_1}^{t_2} (x \,.\, Kx)'dt = 0$$

in view of (3),

$$\int_{t_1}^{t_2} (2x' \,.\, K^\circ x + x \,.\, K'x)\, dt = 0.$$

If this identity is added to (25a) it follows that

$$\int_{t_1}^{t_2} (|x' - (\tfrac{1}{2}G^* - K^\circ)\, x|^2 - x \,.\, Mx)\, dt = 0, \tag{26}$$

where $M = F - K' + (\tfrac{1}{2}G - K^\circ)(\tfrac{1}{2}G^* - K^\circ)$.

Suppose that $M = M(t)$ satisfies the condition $M \leqslant 0$ throughout. By the definition of M, this is the case if $K = K(t)$ satisfies the inequality (25) for $a \leqslant t \leqslant b$. Then, since $M \leqslant 0$ means, by §2, that $y \,.\, M^\circ y \leqslant 0$ or, what is the same thing, that $y \,.\, My \leqslant 0$ holds for every vector y, both terms of the integrand in (26) are non-negative for $t_1 \leqslant t \leqslant t_2$. It follows therefore from (26) that both of these terms, which are continuous functions, must vanish for all t between t_1 and t_2. The vanishing of the first term for $t_1 \leqslant t \leqslant t_2$

implies that the vector $x(t)$ is a solution of the differential equation $x' = A(t)x$, where $A = \frac{1}{2}G^* - K^\circ$. Since this differential equation is of first order, and since its solution $x(t)$ satisfies, by (3), the initial condition $x(t_1) = 0$, it follows from the uniqueness theorem of linear differential equations that $x(t) \equiv 0$.

This shows that if (25) is satisfied, then (3) implies $x(t) \equiv 0$ for every solution of (1). This proves (IV).

9. Another criterion. In what follows, an idea of Lichtenstein **(1**, p. 1298**)**, applied by him to a partial differential equation of (elliptic type) for a scalar, will be adapted to (2), an ordinary differential equation for a vector x.

As in §6, suppose that $G(t)$ is continuously differentiable. Then (2) possesses the adjoint

$$x'' - G^*(t)x' + (F^*(t) - G^{*\prime}(t))x = 0 \tag{27}$$

(in which the coefficient matrices are continuous functions). In fact, if $m(x)$ and $n(x)$ denote the (vector) differential operators on the left of (2) and (27), respectively, then, since $u \,.\, Av = v \,.\, A^*u$, it is seen that

$$y \,.\, m(x) - x \,.\, n(y) = y \,.\, x'' - x \,.\, y'' + y \,.\, Gx' + y' \,.\, Gx + y \,.\, G'x$$

is an identity. Since the expression on the right of this identity is the derivative of $y \,.\, x' - x \,.\, y' + y \,.\, Gx$, the assertion follows.

We do not known an "elementary" verification of the following theorem (the analogue of which was used by Lichtenstein for his purposes **(1**, p. 1298**)**):

(V) *Let $F(t)$ be continuous, and $G(t)$ continuously differentiable, on $a \leqslant t \leqslant b$. Then both* (2) *and* (27) *are disconjugate on $a \leqslant t \leqslant b$ if either* (2) *or* (27) *is disconjugate there.*

The truth of this assertion follows directly from the definition of disconjugate behavior and from the fact that the pair of adjoint differential equations (2), (27) and the (common) boundary condition (3) determine Green kernels which are adjoint (i.e., transposed) kernels in the sense of Fredholm's theory.

Remark. If use is made of (V), then the second of the criteria in (III) follows from the first (and conversely). In fact, if the coefficients of (2) are replaced by those of (27), then (21) goes over into

$$F^* - G^{*\prime} \leqslant -\tfrac{1}{4}(-G^*)(-G^*)^* = -\tfrac{1}{4}G^*G,$$

a condition which can be written in the form (22), since $A \leqslant B$ is equivalent to $A^* \leqslant B^*$ (§2).

10. Convexity. The assumption (21) for the coefficients of (2) implies a geometrical restriction for the squared lengths of solution vectors of (2); cf. **(6)**, where $G = 0$.

LEMMA 3. *If $x(t)$ is any solution of* (2), *then the graph of $r(t) = |x(t)|^2$ in a (t, r)-plane is convex (from below) on every t-interval on which* (21) *is satisfied.*

This assertion is equivalent to the statement that the second derivative of $|x(t)|^2$ is non-negative by virtue of (21). But if $K = 0$, then (25) and

$$r(t) = |x(t)|^2 + \int^t x \,.\, Kx\, dt \tag{28}$$

reduce to (21) and $|x(t)|^2$, respectively. Hence the assertion is contained in the statement that (28) satisfies

$$r'' \geqslant 0 \tag{29}$$

by virtue of (25) and (2). But this can be verified as follows:

Let the expression (28), where $|x|^2 = x \,.\, x$, be differentiated twice, and let x'' be substituted from (2) in the resulting representation of r''. It then follows that $\frac{1}{2}r''$ is the integrand in (26). Hence (29) follows from (25), the latter being equivalent to $x \,.\, Mx \leqslant 0$. Thus Lemma 3 is verified.

11. Bounded and unbounded solutions. This lemma makes it possible to generalize the existence theorem of **(6)**, where

$$F(t) \leqslant 0, \quad G(t) = 0, \tag{30}$$

to the case of an arbitrary pair F, G satisfying (21), as follows:

(VI) *If the coefficients of* (2) *satisfy* (21) *for large positive* t, *then the system* (2) *of order* $2n$ *possesses* n *linearly independent solution vectors, say*

$$x_1(t), \ldots, x_i(t), \ldots, x_n(t), \tag{31}$$

such that the lengths $|x_i(t)|$ *of the vectors* (31) *tend to finite limits* $(\geqslant 0)$ *as* $t \to \infty$, *whereas* $|x(t)| \to \infty$ *holds for the lengths of all solutions vectors which are linearly independent of the* n *particular solutions* (31).

The proof of (VI) will not use (21) directly, but only its consequence, according to which

$$r''(t) \geqslant 0, \text{ where } r = |x|^2, \tag{32}$$

holds for all solutions $x = x(t)$ of (2).

Proof of (VI). The first part of (III) assures that, if t^1 and $t^2 (> t^1)$ are large enough, then (2) is disconjugate on $t^1 \leqslant t \leqslant t^2$. Hence, for the same reasons as in **(6**, p. 363**)**, there belongs to any pair of constant vectors u, v a unique solution $x = x(t)$ of (1) satisfying

$$x(t^1) = u, \quad x(t^2) = v. \tag{33}$$

Without loss of generality, it can be assumed that $t^1 = 0$. Let u be any unit vector (there are ∞^n such choices of u) and, with reference to a fixed choice of u, let $x_k(t)$, where $k = 1, 2, \ldots$, denote the solution vector $x(t)$ belonging to the case $t^1 = 0, t^2 = k, v = 0$ of (33); so that

$$|x_k(0)| = 1, \quad x_k(k) = 0. \tag{34}$$

It is clear from (34) and (32) that the sequence $x_1(t), \ldots, x_k(t), \ldots$ is uniformly bounded on every finite t-interval; in fact,

$$|x_k(t)| \leqslant 1, \qquad 0 \leqslant t \leqslant k. \tag{35}$$

Hence, if the sequence of the derivatives is bounded at $t = 0$, that is, if

$$|x'_k(0)| < \text{const. as } k \to \infty, \tag{36}$$

then the proof of (VI) can be completed in the same way as in **(6**, pp. 364–365**)**. But (36) can be proved as follows:

Since $x'_k(t)$ is continuous, there exists on $0 \leqslant t \leqslant 1$ some t^k satisfying

$$|x'_k(t^k)| = \max_{0 \leqslant t \leqslant 1} |x'_k(t)|. \tag{37}$$

Let γ denote a constant satisfying

$$|F(t)x| \leqslant \gamma|x|, \quad |G(t)x| \leqslant \gamma|x| \qquad (0 \leqslant t \leqslant 1) \tag{38}$$

for every vector x. It can be supposed that $\gamma \geqslant 2$. Then, for every k, there is a number $\tau = \tau_k$ for which

$$n\gamma|\tau| = 1 \text{ and } 0 \leqslant t^k + \tau \leqslant 1. \tag{39}$$

Only the algebraic sign of τ_k (and not $|\tau_k|$) depends on k. By the mean value theorem of differential calculus,

$$x_k(t^k + \tau) = x_k(t^k) + \tau x_k'(t^k) + \tfrac{1}{2}\tau^2 x_k''(\theta), \tag{40}$$

where θ is between t^k and $t^k + \tau$. Actually, (40) is to be considered as a set of n scalar equations, one for each component of x_k, and the value of θ varies from equation to equation but, in each case, is a number between t^k and $t^k + \tau$.

It follows from (2) and (38) that the absolute value of the jth component of $x_k''(\theta)$ does not exceed $\gamma(|x_k'(\theta_j)| + |x_k(\theta_j)|)$, where $\theta = \theta_j$ is an intermediary value belonging to the jth equation in (40). Since $|x(\theta_j)| \leqslant 1$, by (35), and $|x_k'(\theta_j)| \leqslant |x_k'(t^k)|$, by (37), it follows that $|x_k''(\theta)| \leqslant n\gamma(|x_k'(t^k)|+1)$. In view of (35), neither $|x_k(t^k)|$ nor $|x_k(t^k + \tau)|$ exceeds 1. Hence (40) implies that

$$|\tau x_k'(t^k)| \leqslant 2 + \tfrac{1}{2}n\tau^2\gamma(|x_k'(t^k)| + 1)$$

or

$$|\tau|(1 - \tfrac{1}{2}n|\tau|\gamma)|x_k'(t^k)| \leqslant 2 + \tfrac{1}{2}n\tau^2\gamma.$$

From the first part of (39),

$$\tfrac{1}{2}(n\gamma)^{-1}|x_k'(t^k)| \leqslant 2 + \tfrac{1}{2}(n\gamma)^{-1}.$$

Consequently, the sequence $x_1'(t^1), x_2'(t^2), \ldots$ is bounded. In view of (37), this proves (36).

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ON A THEOREM OF LE ROUX

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1. The Theorem of Le Roux. Let $U(x, y, \alpha)$ be any solution of the linear hyperbolic differential equation

$$L(u) \equiv u_{xy}+a(x, y)u_x+b(x, y)u_y+c(x, y)u = 0, \tag{1}$$

containing a parameter α. J. Le Roux **(5)** has shown that the function $u(x, y)$ defined by

$$u(x, y) = \int_{\alpha_0}^{x} U(x, y, \alpha) f(\alpha)\, d\alpha \qquad \alpha_0 = \text{const.} \tag{2}$$

with f arbitrary, is also a solution of (1) provided that

$$\frac{\partial U}{\partial y} + a(x, y)\, U = 0, \qquad \alpha = x. \tag{3}$$

In his proof Le Roux assumes $U(x, y, \alpha)$ is sufficiently regular in the region of interest, so that (2) may be differentiated under the integral sign. However, this last assumption is not fulfilled by the particular functions U used, for example, by Le Roux, Darboux **(2**, pp. 54ff.**)** and others in the theory of the Euler-Poisson equation $E(\beta, \beta')$:

$$u_{xy} - \frac{\beta'}{x-y} u_x + \frac{\beta}{x-y} u_y = 0. \tag{4}$$

In this particular case of (1), as is easily verified, the function

$$U(x, y, \alpha) = (x-\alpha)^{-\beta}\,(\alpha-y)^{-\beta'},$$

which for $\beta > 0$ is singular on $x = \alpha$, is a solution of (4) satisfying (3) (in a limiting sense), and may be used, for $\beta < 1$, to generate other solutions of (4) by means of (2). Integrals (2) with "singular" U occur also in a discussion of Bergmann's integral operator method by Diaz and Ludford **(3)**. Integrals of a similar type again appear in a recent paper by Weinstein **(6)**, which is devoted to the solution of a problem which includes, as special cases, the radiation problem for the wave equation and a problem of Tricomi, Germain, and Bader.

It is therefore of interest to give a general theorem which completes that of Le Roux so as to cover certain cases, such as the one just mentioned, of singular functions $U(x, y, \alpha)$.

The theorem given below can also be considered as an extension of a theorem of Bergmann **(1)** to the case when the kernels involved become singular, but

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the integration is still taken along the real axis. It is possible to base a proof of the present theorem on Bergmann's theorem, but then the simplicity of Le Roux's original idea of employing the classical superposition principle of linear equations is obscured. The generalisations discussed at the conclusion of this paper apparently have no counterpart in Bergmann's theory.

2. An Extension of the theorem of Le Roux. *Let R be an open region of the x, y plane where the coefficients of* (1) *are continuous, and let $U(x, y, \alpha) = \tilde{U}(x, y, \alpha)/(x-\alpha)^{\lambda}$, with $0 < \lambda < 1$, be a solution of $L(u) = 0$ for all (x, y) in R, wherever $\alpha_0 \leqslant \alpha < x$. Also let U be such that $\tilde{U}$, $\tilde{U}_x$, $\tilde{U}_y$, $\tilde{U}_{xy}$, $\tilde{U}_{\alpha}$, $\tilde{U}_{\alpha x}$, $\tilde{U}_{\alpha y}$, $\tilde{U}_{\alpha xy}$ are continuous on the three dimensional region consisting of all (x, y, α) with (x, y) in R and $\alpha_0 \leqslant \alpha \leqslant x$. Then*

$$u(x, y) = \int_{\alpha_0}^{x} \frac{\tilde{U}(x, y, \alpha)}{(x-\alpha)^{\lambda}} f(\alpha)\, d\alpha \tag{5}$$

is, for an arbitrary (once continuously differentiable) function $f(\alpha)$, a solution of (1) *in R.*

Proof. It suffices to consider $f(\alpha) \equiv 1$, since otherwise $f(\alpha)$ may be absorbed into $\tilde{U}$. Notice first that the Le Roux condition (3) is apparently missing from the statement of the theorem. However, on substitution into (1), one obtains (for $\alpha \neq x$)

$$L(U) = L(\tilde{U}/(x-\alpha)^{\lambda}) = \frac{1}{(x-\alpha)^{\lambda+1}}[(x-\alpha)\, L(\tilde{U}) - \lambda(\tilde{U}_y + a\tilde{U})] = 0. \tag{6}$$

Therefore, since $\lambda \neq 0$,

$$\tilde{U}_y + a\tilde{U} = \frac{x-\alpha}{\lambda} L(\tilde{U}), \qquad \alpha \neq x. \tag{7}$$

But each side of this equation is, from the hypotheses made about $\tilde{U}$, continuous as $\alpha \to x$. Hence $\tilde{U}_y + a\tilde{U}$ vanishes at least as fast as $(x-\alpha)$ as $\alpha \to x$, which implies that $U_y + aU$ vanishes at least as fast as $(x-\alpha)^{1-\lambda}$, or that (3) is satisfied in the limit as $\alpha \to x$, i.e.,

$$\lim_{\alpha \to x} \left[\frac{\partial U}{\partial y}(x, y, \alpha) + a(x, y)\, U(x, y, \alpha) \right] = 0.$$

Before differentiating, it is necessary to integrate (5) by parts, to obtain, with $f(\alpha) \equiv 1$,

$$u(x, y) = \frac{(x-\alpha_0)^{1-\lambda}\, \tilde{U}(x, y, \alpha_0)}{1-\lambda} + \int_{\alpha_0}^{x} \frac{(x-\alpha)^{1-\lambda}}{1-\lambda} \tilde{U}_{\alpha}(x, y, \alpha)\, d\alpha.$$

With $u(x, y)$ in this form, the partial derivatives u_x, u_y, u_{xy} can be computed by direct differentiation under the integral sign, and

$$L(u) \equiv \frac{(x-\alpha_0)^{1-\lambda}}{1-\lambda} \left[L(\tilde{U}) + \frac{1-\lambda}{x-\alpha_0}(\tilde{U}_y + a\tilde{U}) \right]_{\alpha=\alpha_0} + \int_{\alpha_0}^{x} \frac{(x-\alpha)^{1-\lambda}}{1-\lambda} \left[\frac{\partial L(\tilde{U})}{\partial \alpha} + \frac{1-\lambda}{x-\alpha} \cdot \frac{\partial}{\partial \alpha}(\tilde{U}_y + a\tilde{U}) \right] d\alpha.$$

But, from (7),

$$L(\bar{U}) = \frac{\lambda}{x-\alpha}(\bar{U}_y + a\bar{U}), \qquad \alpha \neq x,$$

so that

$$\begin{aligned} L(u) &= \frac{(x-\alpha_0)^{-\lambda}}{1-\lambda}[\bar{U}_y + a\bar{U}]_{\alpha=\alpha_0} \\ &\quad + \int_{\alpha_0}^{x} \frac{(x-\alpha)^{1-\lambda}}{1-\lambda}\left[\lambda\frac{\partial}{\partial\alpha}\left(\frac{\bar{U}_y + a\bar{U}}{x-\alpha}\right) + \frac{1-\lambda}{x-\alpha}\frac{\partial}{\partial\alpha}(\bar{U}_y + a\bar{U})\right]d\alpha \\ &= \frac{(x-\alpha_0)^{-\lambda}}{1-\lambda}[\bar{U}_y + a\bar{U}]_{\alpha=\alpha_0} + \int_{\alpha_0}^{x}\frac{\partial}{\partial\alpha}\left[\frac{(x-\alpha)^{-\lambda}}{1-\lambda}(\bar{U}_y + a\bar{U})\right]d\alpha \\ &= 0, \end{aligned}$$

since $\lim_{\alpha\to x}[U_y + aU] = 0$.

3. Generalisations. The theorem is more general than would appear from the special form of U employed. For, in any region where the coefficients a, b, c of (1) are regular in the sense of Hadamard (4, p. 11), a theorem of Le Roux and Delassus (4, p. 72) shows that any singular curve of a singular solution $U(x, y, \alpha)$ of (1)—certain assumptions being made as to the nature of the singularity—must be a characteristic of (1), whose position, in the present case, will depend upon α, that is, $x = c(\alpha)$ or $y = \bar{c}(\alpha)$. Assuming an algebraic type of singularity on such curves, a fairly general definite integral, with variable limits, to be considered as a possible solution of (1), is

$$(8) \qquad u(x, y) = \int_{\alpha_0}^{\theta(x,y)} \frac{\bar{U}(x, y, \alpha)\, f(\alpha)}{[x - c(\alpha)]^{\lambda}[y - \bar{c}(\alpha)]^{\lambda'}}\, d\alpha, \qquad 0 < \lambda, \lambda' < 1,$$

where $\theta(x, y) > \alpha_0 = \text{const.}$ for (x, y) in R. Suppose that the curve $c^{-1}(x) = \bar{c}^{-1}(y)$ does not intersect[1] R, that the integrand in (8) is a one parameter family of solutions of (1) with $\bar{U}$ regular, in a sense analogous to that in the theorem, and that (8) is to be a solution of (1). For (x, y) in any given sufficiently small neighborhood lying in R and not intersecting either $c^{-1}(x) = \theta(x, y)$ or $\bar{c}^{-1}(y) = \theta(x, y)$, the integral in (8) is easily seen to be the sum of integrals of the form

$$(9) \qquad \begin{cases} \displaystyle\int_{\alpha_1}^{\theta(x,y)} \bar{U}_1(x, y, \alpha)\, f(\alpha)\, d\alpha, \\ \displaystyle\int_{\alpha_2}^{x} \frac{\bar{U}_2(x, y, \alpha)}{(x-\alpha)^{\lambda}} f(\alpha)\, d\alpha, \\ \displaystyle\int_{\alpha_3}^{y} \frac{\bar{U}_3(x, y, \alpha)}{(y-\alpha)^{\lambda'}} f(\alpha)\, d\alpha, \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3$ are suitable constants, and $\bar{U}_1$, $\bar{U}_2$, $\bar{U}_3$ are sufficiently regular (see theorem) in their respective domains of definition. On forming $L(u)$ from

[1] Although the curve may be part of the boundary of R.

(8) the integrals of the last two types in (9) give zero, according to the theorem. The remaining integral is of the form discussed by Le Roux, who showed that $\theta(x, y)$ must be either a function of x alone or of y alone if the operation L is to produce zero. Thus our theorem shows that if (8) is to be a solution of (1), then $\theta(x, y)$ must be a function of x or y alone, and the Le Roux condition corresponding to (3) is to be satisfied.

Similar considerations apply to definite integrals with both lower and upper limits of integration variable, such as were discussed by Le Roux; and also for λ or λ' zero in (8).

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UNIQUENESS IN BOUNDARY VALUE PROBLEMS FOR THE SECOND ORDER HYPERBOLIC EQUATION

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Introduction. We study linear normal hyperbolic partial differential equations of the second order, with one dependent variable u, and N independent variables x^i $(i = 1, \ldots, N)$. The uniqueness theorem connected with the Cauchy problem for this type of equation is well known and in effect states that if u and its first normal derivatives vanish on a spacelike initial surface S then u vanishes in a certain conical region which contains S **(1**, p. 379**)**. In the present work we also envisage a timelike cylindrical surface T which meets S in a rim C of N-2 dimensions, and we assign a single homogeneous boundary condition, of the type familiar from potential theory, on T. The homogeneous Cauchy conditions are also assumed on that part of S which is inside T. We shall prove that the solution then vanishes identically in the region inside T. If the homogeneous boundary condition is given for a certain "time interval" along T, the proof shows that u vanishes in this same interval of the timelike variable.

The boundary conditions considered are of the Dirichlet, Neumann, and Robin type. Uniqueness with the Dirichlet condition has been proved by Hörmander **(4)** using a different approach by which estimates of the solution in the non-homogeneous case can be found. The method used here is also applied to systems of second order normal hyperbolic equations having similar second order terms.

1. Geometric background. Let the differential equation be written in the form

$$L[u] = a^{ik} \frac{\partial^2 u}{\partial x^i \partial x^k} + \beta^i \frac{\partial u}{\partial x^i} + cu = 0. \tag{1.1}$$

Here the summation convention for repeated indices is understood, i and k ranging from 1 to N. The coefficients in the equation are assumed to be four times continuously differentiable (C^4) functions of the x^i. The signature of the quadratic form $a^{ik}\,\xi_i\xi_k$ is taken to be (1, N-1), so that the equation is normal hyperbolic. We may suppose $a^{ik} = a^{ki}$ without loss of generality.

The geometrical aspect is best treated by means of the Riemannian geometry associated with the coefficients a^{ik}; these latter have under coordinate changes the transformation law of a contravariant tensor. Since (1.1) is hyperbolic, the determinant $|a^{ik}|$ is not zero; hence the matrix (a^{ik}) has an inverse (a_{ik}).

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This definition of the a_{ik}, which in explicit form reads

$$a^{ik}a_{kj} = \delta^i{}_j$$

shows that a_{ik} and a^{ik} are associate tensors, a_{ik} being covariant and symmetric.

The Riemannian metric associated with (1.1) is determined by the line-element

$$ds^2 = a_{ik}\,dx^i\,dx^k,$$

which also has the signature $(1, N-1)$. The measure of lengths and angles determined by this metric will be used throughout this paper. A vector v^i is timelike if its square length

$$v^2 = a_{ik}v^iv^k = a^{ik}v_iv_k$$

is positive, and spacelike if v^2 is negative. The vector is null if $v^2 = 0$. The normal vector to a surface $\phi = \text{const.}$ has contravariant components $a^{ik}\partial\phi/\partial x^k$, and the surface is spacelike if its normal is timelike and vice versa. The surface is null if its normal vector is null; and is then a characteristic surface of the hyperbolic differential equation. To each point P with coordinates x^i there is associated a null or characteristic conoid with vertex P.

According to the theory of invariant differential operators in Riemannian spaces, the operator L can be written in the form

$$L[u] = \Delta u + b\cdot\nabla u + cu,$$

where ∇u denotes the gradient vector, where c is a scalar invariant and b a vector; and where

$$\Delta u = \frac{1}{\sqrt{a}}\frac{\partial}{\partial x^i}\left(\sqrt{a}\,a^{ik}\frac{\partial u}{\partial x^k}\right)$$

is the Laplacian of u (**2**, §4.2).

The timelike cylindrical hypersurface T upon which the boundary conditions shall be assigned is understood to be topologically equivalent to the product of a compact closed $(N-2)$-dimensional manifold (spacelike) and a line (timelike). By a spacelike cross-section of T, or surface spanning T, is meant a spacelike surface which divides the region interior to T into two separated regions. With respect to the spanning surface, these correspond to the past and future. We assume that T does not touch or intersect itself and therefore that the interior region is well defined. The equation defining T in terms of the given coordinates shall be four times continuously differentiable.

Though we may regard T as extending far into the past and future, we shall in practice work with a finite length of it defined in the following way. Let R denote the region interior to T which is covered by a 1-parameter family of spacelike surfaces Σ_t spanning T. If S is a given spacelike initial surface, we may take for R the region covered by surfaces Σ_t geodesically parallel to S. Therefore such a region R always exists, given S, though it may be of finite

extent. If the space is flat, there exists a family Σ_t of parallel planes such that R extends arbitrarily far into the future.

For simplicity we assume that the initial surface S is Σ_0. It is actually sufficient if the rim $C = S \cap T$ should lie in the closure of R, since we can suppose that the solution function vanishes identically in R "before" or "below" S.

Then the problem to be studied may be formulated as follows. Let

$$L[u] = 0 \quad \text{in } R; \tag{1.2}$$

and let the Cauchy data for u on S vanish:

$$u = 0, \ \frac{\partial u}{\partial n} = 0 \quad \text{on } S. \tag{1.3}$$

Let u satisfy on T either

$$u = 0 \quad \text{or} \quad \frac{\partial u}{\partial n} = 0. \tag{1.4}$$

We show that u vanishes in R. We may assume that u vanishes in the interior of T below $S = \Sigma_0$.

From the theory of Cauchy's problem it is known that (1.2) and (1.3) ensure the vanishing of u in the domain of dependence D_t upon S (see **1**, p. 310; **2**, p. 62; **3**, ch. 9 §5). This domain D_t is bounded by two characteristic surfaces each containing the rim $C = T \cap S$. We shall first establish the result for a second family of surfaces S_t constructed in the following section, which may exist only over small timelike intervals. This will demonstrate that if (1.3) hold on Σ_t, then u will vanish on Σ_τ, for $t \leqslant \tau \leqslant t + \epsilon$, where ϵ may depend on t. From the Heine-Borel theorem it will follow that (1.2), (1.3), and (1.4) ensure the vanishing of u throughout R.

At each step of this process we know that u vanishes in the domain of dependence on Σ_t. That is, u vanishes on $\Sigma_{t+\epsilon}$ except for a strip of width of order ϵ adjoining the boundary $\Sigma_{t+\epsilon} \cap T$ of $\Sigma_{t+\epsilon}$. We have therefore to prove that u vanishes in this strip.

2. A coordinate system. Let the initial surface S cut T in the rim $C = C_0$ of $N - 2$ dimensions. We now denote by S_0 the portion of S bounded by T.

Let $T(n)$ denote the timelike surface geodesically parallel to T at inward distance n. In drawing this surface we first select a compact region of T, such as the portion of T intercepted by Σ_{-t} and Σ_t. The parallel surfaces generated by this region of T are then defined for n sufficiently small, say $0 \leqslant n \leqslant n_0$. and are of class C^3 in the x^i. The $T(n)$ are timelike since their normals are spacelike geodesics (**1**, p. 365; **3**, p. 45).

Now let timelike geodesics g_T of the cylinder subspace T be drawn from each point of C_0, orthogonal to C_0 in T. The locus of points at subspace geodesic

distance t from C_0 is an $N-2$ dimensional surface C_t, geodesically parallel to C_0 in T. The g_T are also orthogonal to C_t in T, and hence also when both are regarded as loci in the full space-time (3, p. 69).

At each point of C_t let the spacelike geodesics, normal to T, in the full space, be drawn inward; these have length n_0. The geodesics drawn from C_t generate a C^2 surface S_t which is topologically the product of C_t by an interval. Let S_t meet $T(n_0)$ in the inner rim $C_t(n_0)$.

Provided that n_0 is chosen sufficiently small, and that a compact t-interval is considered, the surfaces S_t will be spacelike. To verify this, we note that on T the normals to S_t coincide in direction with the timelike sub-geodesics g_T. Since the direction of the normal varies continuously on S_t, there is an n-interval of positive length $n(t)$ whereon S is spacelike. We have therefore to choose $n_0 < \min n(t)$.

We now assert that for sufficiently small t, the inner rim $C_t(n_0)$ lies in D_S. We note, again presuming n_0 was chosen sufficiently small in the first place, that $C_0(n_0)$ lies in the interior of D_S. Since $C_t(n_0)$ varies continuously with t, the statement is valid for some t-interval, say $0 \leqslant t \leqslant t_0$. Consequently u and its derivatives will vanish identically on $C_t(n_0)$ in this interval.

Coordinates $\xi_1, \ldots, \xi_{N-1}, t$ of the points on S_t will be assigned as follows. Let $\xi_1, \ldots, \xi_{N-2}$ be coordinates parametrizing C_0. Given P on S_t, and $T(n)$, follow the spacelike geodesic of S_t through P back to C_t on T, and then follow the subspace geodesic g_T along T to C_0, meeting C_0 at $\xi_1, \ldots, \xi_{N-2}$, say. We then take $\xi_1, \ldots, \xi_{N-2}, \xi_{N-1} = n$ and t as coordinates of P on S_t.

It is easily shown that the g_T are C^2 curves (with respect to the x^i), and that the geodesics of S_t are C^2. It follows that the (ξ_r, t) coordinates are related to the (x^i) coordinates by a C^2 transformation. Hence also the components a^{ik} of the metric tensor, expressed in the system, are C^1 functions of ξ_r and t.

On the boundary surface T, the parametric lines of the $\xi_\alpha, \alpha = 1, \ldots, N-1$ are orthogonal to the parametric lines of t. To show this, we recall that the g_T, which are the parametric lines of t on T, are the orthogonal trajectories of the C_t. This shows the statement is true for $\xi_1, \ldots, \xi_{N-2}$. Finally, the parametric lines of $n = \xi_{N-1}$ are the geodesics normal to T, and therefore also orthogonal to the g_T. This shows that the parametric lines of ξ_{N-1} are orthogonal to the t-lines as well. It follows that in the (ξ_r, t) coordinates the components a_{rN} $(r = 1, \ldots, N-1)$ of the metric tensor vanish on T. Therefore the contravariant components $a^{rN} (r = 1, \ldots, N-1)$ which appear as coefficients in the differential equation, also vanish there, as a brief computation shows. Thus, on T,

$$a^{rN} = 0, \qquad r \neq N. \tag{2.1}$$

3. The basic inequality. In the (ξ^α, t) coordinate system the differential equation can be written

$$L[u] = a^{NN}u_{tt} + a^{rs}u_{\xi^r\xi^s} + 2a^{Nr}u_{\xi^r} + \beta^r u_{\xi^r} + \beta^N u_t + cu = 0, \tag{3.1}$$

where summation over r and s runs from 1 to $N-1$. The quadratic form $Q_0(x) = a^{rs}x_r x_s$ is negative definite on account of the signature of the overall metric. Also $a^{NN} > 0$. Let us therefore define a positive definite metric in the subspace S_t: $t = \text{const.}$ by writing

$$g^{rs} = -\frac{a^{rs}}{a^{NN}}. \tag{3.2}$$

We may define the associate tensor g_{rs} in the usual way since $|g^{rs}| \neq 0$. The volume element dS_t is $\sqrt{g}\, d\xi_1 \ldots d\xi_{N-1}$ where $g = |g_{rs}|$.

Let us denote by ∇_t and Δ_t the gradient and Laplacian operators in the subspace S_t. Then the differential equation takes the form

$$u_{tt} = \Delta_t u + b_t \cdot \nabla_t u + c_1 u + \beta_1 u_t + \sum_{r=1}^{N-1} \beta_1^r u_{t\xi^r}, \tag{3.3}$$

where $(b_t)^r = -\beta^r/a^{NN}$, $c_1 = -c/a^{NN}$, $\beta_1 = -\beta^N/a_{NN}$, and

$$\beta_1^r = -2a^{Nr}/a^{NN}. \tag{3.4}$$

We note that the vector $\beta_1{}^r$ vanishes on the rim C_t.

We consider the positive definite integral

$$E(t) = \int_{S_t} \{u_t^2 + (\nabla_t u)^2\}\, dS_t, \tag{3.5}$$

which vanishes only if all first derivatives of u with respect to ξ^r and t vanish on S_t. Now, differentiating with respect to t, we find

$$\frac{dE(t)}{dt} = 2\int_{S_t} \Big\{ u_t u_{tt} + \nabla_t u \cdot \nabla_t u_t + C^{rs} u_{\xi^r} u_{\xi^s} + [u_t^2 + (\nabla_t u)^2] \frac{\partial \log |g|^{\frac{1}{2}}}{\partial t} \Big\}\, dS_t. \tag{3.6}$$

Here the C^{rs} contain partial derivatives of the g^{rs} with respect to t. Replacing u_{tt} by its expression in (3.2), we have

$$\frac{dE(t)}{dt} = 2\int_{S_t} \{u_t \Delta_t u + \nabla_t u \cdot \nabla_t u_t + u_t b_t \cdot \nabla_t u + c_1 u u_t + \beta_2 u_t^2 + \beta^r u_{t\xi^r} u_t + C_2{}^{rs} u_{\xi^r} u_{\xi^s}\}\, dS_t. \tag{3.7}$$

Here summation over r, s from 1 to $N-1$ is understood, while β_2 and the $C_2{}^{rs}$ are new coefficients, of the same type as β_1 and C^{rs}, which include the terms arising from the derivative of $g(t)$.

The terms in the integrand of (3.7) will be separated into three sets. To the first two terms we apply the Gauss theorem for the domain S_t with boundary $C_t - C_t(n_0)$, finding

$$\int_{S_t} \{u_t \Delta_t u + \nabla_t u_t \cdot \nabla_t u\}\, dS_t = \int_{S_t} \nabla_t \cdot \{u_t \nabla_t u\}\, dS_t = \int_{C_t - C_t(n_0)} u_t \frac{\partial u}{\partial n}\, ds_t. \tag{3.8}$$

Here ds_t is the "surface element" on C_t and $C_t(n_0)$, while $\partial/\partial n$ denotes the normal derivative in S_t across C_t or $C_t(n_0)$. However on C_t the direction of this differentiation is also normal to T in the full space, and is consequently the normal to T in the full space. That is, the derivative $\partial u/\partial n$ in (3.8) is the normal derivative in the sense of the given boundary value problems. We also note that $C_t(n_0)$ lies in D_S for $0 \leqslant t \leqslant t_0$, and since u vanishes identically in D_S the integral over $C_t(n_0)$ drops out, provided $t < t_0$. Thus the integrals of the first two terms on the right are together equal to

$$\int_{C_t} u_t \frac{\partial u}{\partial n} ds_t.$$

The term containing the mixed second derivatives may be transformed to an integral containing only first derivatives. We have, applying the Gauss theorem on S_t to the vector $u_t^2\beta_1^r$,

$$\int_{C_t - C_t(n_0)} \beta_{1n} u_t^2 ds_t = \int_{S_t} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^r}(\sqrt{g}\, u_t^2 \beta_1^r)\, dS_t$$
$$= \int_{S_t} u_t^2 \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^r}(\sqrt{g}\, \beta_1^r)\, dS_t + 2\int_{S_t} \beta_1^r u_t u_{t\xi^r} dS_t.$$

Here the integral over $C_t(n_0)$ vanishes as before, and now the integral over C_t drops out since all the components of the "vector" β_1^r vanish on C_t. We then obtain

$$\int_{S_t} \beta_1^r u_t u_{t\xi^r} dS_t = -\tfrac{1}{2}\int_{S_t} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^r}(\beta_1^r \sqrt{g})\, u_t^2 dS_t. \tag{3.9}$$

From (3.7), (3.8) and (3.9) we find

$$\frac{dE(t)}{dt} = 2\int_{C_t} u_t u_N ds_t$$
$$+ 2\int_{S_t} \{u_t b^r u_{t\xi^r} + cuu_t + \beta_2 u_t^2 + C_1^{rs} u_{t\xi^r} u_{t\xi^s}\}\, dS_t,$$

where β_2 incorporates the term (3.9). Here the integral over S_t is a quadratic functional of u and the first derivatives of u. We integrate this equation from 0 to t, noting that $E(0) = 0$ owing to the homogeneous Cauchy conditions on S_0. Thus

$$E(t) = 2\int_0^t d\tau \int_{C_\tau} u_\tau u_N ds_\tau$$
$$+ 2\int_0^t d\tau \int_{S_\tau} \{u_\tau b^r u_{\xi^r} + cuu_\tau + \beta_2 u_\tau^2 + C_1^{rs} u_{\xi^r} u_{\xi^s}\}\, dS_\tau. \tag{3.10}$$

The second term on the right is again a quadratic functional of u and its first derivatives. Now if, as we shall assume, these vanish for $t = 0$, we can follow the method (**1**, p. 310) used in the uniqueness theorem for the Cauchy problem, and find an estimate for this quantity in terms of $E(t)$. The calculation

is straightforward, and shows that the second term on the right is less than

$$K \int_0^t E(\tau)\, d\tau,$$

for some constant K.

Let us now assume that the surface integral, the first term on the right of (3.10), is non-positive. Then we have

$$E(t) \leqslant K \int_0^t E(\tau)\, d\tau,$$

and on integrating from 0 to α, say, we find

$$\int_0^\alpha E(t)\, dt \leqslant \alpha \max E(t) \leqslant K\alpha \int_0^\alpha E(\tau)\, d\tau.$$

Thus either $K\alpha \geqslant 1$, or else $E(t) \equiv 0$ for $0 \leqslant t \leqslant \alpha$. The first alternative is surely false for $\alpha < K^{-1}$, hence we conclude that $E(t) \equiv 0$, $0 \leqslant t < K^{-1}$. That is, all first derivatives of u vanish in the region bounded by T, S, and $S_{K^{-1}}$. Since $u = 0$ on S, $u \equiv 0$ in the region.
This establishes the following

LEMMA. *If*

$$J(t) = \int_0^t d\tau \int_{C_\tau} u_\tau u_n ds_\tau \leqslant 0,$$

for $0 \leqslant t \leqslant t_1$, *then* $u \equiv 0$ *in the region covered by* S_t, $0 \leqslant t \leqslant t_1$.

4. Uniqueness for the Dirichlet and Neumann problems. The uniqueness in the large follows from this Lemma by an application of a "Heine-Borel" argument. If Σ_t is a surface of the family covering R, and if u has vanishing Cauchy data on Σ_t, then we may take Σ_t as the S_0 of the Lemma. Assuming that $J(t) \leqslant 0$, we see that u vanishes in a t-neighbourhood of Σ_t. Now u has zero Cauchy data on Σ_0; and so vanishes in a t-neighbourhood of Σ_0. If now

$$t_0 = \underset{t_1 \neq 0}{\text{g.l.b.}}\ t_1,$$

we see that $t_0 > t$ for every t^1 of R. Hence $u \equiv 0$ in R.

Any boundary condition which ensures that the integrand $u_n u_\tau$ of $J(t)$ in the Lemma is non-positive will lead to a uniqueness theorem. For the Dirichlet boundary $u = 0$ on T we have by differentiation, tangential to T, $u_\tau \equiv 0$, and hence $J(t) \equiv 0$. Similarly for the Neumann condition $u_n = 0$. Indeed it is sufficient if $u = 0$ in a certain open subset of T and if $u_n = 0$ on the complementary part of T. Thus we have

THEOREM I. *Let R be a region covered by spacelike surfaces Σ_t and bounded by $S = \Sigma_0$, Σ_{t_1}, and the timelike surface T. Let $L[u] = 0$ in R, let $u \in C^2$ in R and let u have vanishing Cauchy data on S. If then either*

$$u = 0 \quad \text{or} \quad u_n = 0 \qquad \text{on } T$$

then

$$u \equiv 0 \qquad \text{in } R.$$

5. Uniqueness for the Robin problem. The boundary condition here is

$$\frac{\partial u(p)}{\partial n} + h(p)\, u(p) = 0, \qquad p \in T. \tag{5.1}$$

We shall assume that $h(p)$ is C^2 in the original coordinates, and show that a function $k(P)$ can be constructed, so that

$$v(P) = k(P)\, u(P) \tag{5.2}$$

satisfies a similar hyperbolic equation

$$L_1[v] = \Delta v + b_1 \cdot \nabla v + c_1 v = 0, \tag{5.3}$$

together with the Neumann boundary condition

$$\frac{\partial v}{\partial n} = 0. \tag{5.4}$$

From the theorem just proved will then follow the vanishing of v, and hence u, in R.

We now construct the function $k(P)$, requiring $k(P) > 0$, $k(P) \in C^2$. Let n denote inward normal distance from T, and let p_0 be the foot of the normal geodesic to T which passes through a point P in the interior of T, sufficiently close to T. For $0 \leqslant n \leqslant \frac{1}{2}n_0$ we set

$$k(P) = k_0(P) = e^{h(p_0)n} > 0. \tag{5.5}$$

For $n \geqslant n_0$ we set $k(P) \equiv 1$ and for $\frac{1}{2}n_0 \leqslant n \leqslant n_0$ we construct the function

$$k(P) = k_0(P)\, \rho\left(\frac{n}{n_0}\right) + 1 - \rho\left(\frac{n}{n_0}\right). \tag{5.6}$$

Here $\rho(x) \in c^\infty$ shall be non-negative, equal to 1 for $x \leqslant \frac{1}{2}$, and equal to zero for $x \geqslant 1$.

In the neighbourhood of T we have, then,

$$v(P) = e^{h(p_0)n} u(P),$$

so

$$\frac{\partial v(p_0)}{\partial n} = e^{h(p_0)n} \left\{ \frac{\partial u(p_0)}{\partial n} + h(p_0)\, u(p_0) \right\} = 0.$$

That is, $v(P)$ satisfies a homogeneous Neumann condition on T. If u has vanishing Cauchy data on an initial surface S then $u \equiv 0$ in the domain of dependence D_s. Thus $v \equiv 0$ in D_s and so v has vanishing Cauchy data on S as well.

The differential equation satisfied by v is found by straightforward calculation to be (5.3), where

$$b_1 = -2 \nabla \log k(P) + b,$$

and

$$c_1 = k(P)\, L[1/k(P)].$$

Since $k(P) \geqslant \delta > 0$ these coefficients are bounded, and continuous. The proof of Theorem I requires only these properties of these coefficients.

This establishes the uniqueness theorem for the boundary condition of the third kind.

THEOREM II. *Let S and T satisfy the conditions of Theorem I, and let R be defined as above. Let $u \in C^2$, $L[u] = 0$ in R, and let*

$$\frac{\partial u(p)}{\partial n} + h(p)\, u(p) = 0, \qquad p \in T$$

where $h(p) \in C^2$. Then $u(P) \equiv 0$ in R.

This result is also seen to be valid if $u \equiv 0$ on an open subset of T while the above Robin boundary condition holds on the complementary portion of T. It is seen that no condition of positivity for $h(p)$ is required.

6. Systems with similar second order terms. The preceding theorems hold also for systems of linear normal hyperbolic second order equations in which the principal parts of all equations are similar. Consider the system

$$\Delta u_{(m)} + \sum_n b_{mn} \cdot \nabla u_{(n)} + \sum_n c_{mn} u_{(n)} = 0, \tag{6.1}$$

wherein m and n are indices of enumeration for the M dependent variables $u_{(m)}$. Also b_{mn} are an array of vectors, and c_{mn} of scalars, which we assume to be C^1 and C, respectively. Here Δ has its previous meaning. For this system, the uniqueness for the Cauchy problem has in effect been proved in **(2**, pp. 62–64**)**, since the argument given there applies to any system of the form (6.1).

The coordinate system of §2 being taken as before, we may repeat the calculations of §3 leading to the Lemma. We write the differential equations in the form

$$u_{(m)\,tt} = \Delta_t u_{(m)} + \sum_n B_{mn} \cdot \nabla_t u_{(n)} + \sum_n c_{mn} u_{(n)} + \sum_n \beta_{mn} u_{(n)\,t} + \beta_1^r u_{(m)\,t\xi^r}. \tag{6.2}$$

Defining

$$E_{(m)}(t) = \int_{S_t} \{u_{(m)\,t}^2 + (\nabla_t u_{(m)})^2\}\, dS_t, \tag{6.3}$$

we find on differentiating with respect to t that

$$\frac{1}{2}\frac{dE_{(m)}(t)}{dt} = \int_{S_t} \{u_{(m)\,t} \Delta_t u_{(m)} + \nabla_t u_{(m)} \nabla_t u_{(m)\,t} + \sum_n \bar{B}_{mn} \cdot \nabla_t u_{(n)} u_{(m)\,t} + \sum_n \bar{C}_{mn} u_{(n)} u_{(m)\,t} + \sum_n \bar{\beta}_{mn} u_{(n)\,t}^2 + \beta_1^r u_{(m)\,t} u_{(m)\,t\xi^r}\}\, dS_t. \tag{6.4}$$

The first two terms on the right again lead to the surface integral

$$\int_{C_t - C_t(n_0)} u_{(m)\,t} \frac{\partial u_{(m)}}{\partial n}\, dS_t, \tag{6.5}$$

while the last term, containing the $u_{(m)\tau\xi^\tau}$, yields an integral containing only first derivatives, namely

$$-\frac{1}{2}\int_{S_t} u^2_{(m)\,t} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^\tau}(\sqrt{g}\,\beta_1^\tau)\, dS_t.$$

We therefore find

$$\frac{dE_{(m)}}{dt} = Q_{(m)}(u_{(m)},\ u_{(m)\xi^\tau},\ u_{(m)\,t}), \tag{6.6}$$

where $Q_{(m)}$ denotes a quadratic integral expression over S_t. Again, integrating, we have

$$E_{(m)}(t) = \int_0^t Q_{(m)} d\tau \tag{6.7}$$

and it follows easily that the integral on the right is dominated by the expression

$$K\int_0^t \sum_{(m)} E_{(m)}(\tau)\, d\tau.$$

We therefore find

$$E(t) = \sum_m E_{(m)}(t) \leqslant MK \int_0^t E(\tau)\, d\tau, \tag{6.8}$$

provided only that for the sum of the surface integrals we have

$$J(t) = \sum_m \int_0^t d\tau \int_{C_\tau} u_{(m)\tau} \frac{\partial u_{(m)}}{\partial n}\, dS_\tau \leqslant 0. \tag{6.9}$$

From (6.8) follows $E(t) \equiv 0$ for $0 \leqslant t \leqslant t_1$, say, and the uniqueness in the large follows as before. The condition (6.9) is satisfied if either

$$u_{(m)} = 0 \text{ or } \frac{\partial u_{(m)}}{\partial n} = 0 \qquad \text{on } T.$$

Indeed $u_{(m)}$ may vanish in an open set $T_{(m)}$ and $\partial_{u(m)}/\partial n$ in the complementary set $T - T_{(m)}$; and the result holds.

Again, we may replace the Neumann condition by the Robin condition

$$\frac{\partial u_{(m)}}{\partial n} + h_{(m)} u_{(m)} = 0, \qquad h_{(m)} \in C^2,$$

by setting

$$v_{(m)}(P) = k_{(m)}(P)\, u_{(m)}(P),$$

as in (5.2). We verify at once that the $v_{(m)}(P)$ also satisfy a system of the form (6.1), and that

$$\frac{\partial v_{(m)}(p_0)}{\partial n} = e^{h_{(m)}(p_0)\pi}\left(\frac{\partial u_{(m)}}{\partial n} + h_{(m)}u_{(m)}\right)$$
$$= 0.$$

Thus the $v_{(m)}$ vanish identically under the boundary conditions discussed above.

We state these results as follows:

THEOREM III. *Let S, T and R satisfy the conditions of Theorem I, and let $u_{(m)}$ be solutions of the system* (6.1) *with zero Cauchy data on Σ_0. Let*

$$u_{(m)} = 0 \qquad (m = 1, \ldots, M),$$

on an open subset $T_{(m)}$ of T, and let

$$\frac{\partial u_{(m)}}{\partial n} + h_{(m)}u_{(m)} = 0$$

on the complementary subset $T - T_{(m)}$. Then

$$u_{(m)} = 0$$

in R.

An important special case of systems such as (6.1) is the equation $\Delta\phi = 0$ of generalized potential theory. The above result applies to the Dirichlet problem for these differential forms. However, the Neumann problem for this equation involves tangential derivatives in spacelike directions on T, and so is not amenable to this method. In fact, uniqueness does not hold for this Neumann problem.

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ASYMPTOTIC SOLUTION OF DIFFERENTIAL EQUATIONS IN A DOMAIN CONTAINING A REGULAR SINGULAR POINT

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1. Introduction. In this paper we study the asymptotic behavior in λ of the solutions about the origin in the z-plane of the differential equation

$$z^2 \frac{d^2u}{dz^2} - \lambda^2 P(z, \lambda)\, u = 0.$$

Both the variable z and the parameter λ are complex. The coefficient $P(z, \lambda)$ is assumed to be analytic and single-valued in λ at infinity and in z throughout a bounded, closed, simply connected domain D containing $z = 0$. The important restriction is imposed[1] that $P(z, \lambda) \neq 0$ for $z \in D$ when $|\lambda| > N$.

Asymptotic series in descending powers of λ are obtained, which are formal solutions of the differential equation, and it is shown that their partial sums are asymptotic representations for the solutions of this equation. The coefficients in these series are determined recursively in terms of the Taylor series of $P(z, \lambda)$ about $\lambda = \infty$. It is to be noted that the first term of each series may be found from results of Cashwell **(3)**, who considered the differential equation in a somewhat different form. We remark that the equation which we are considering is exceptional in the theory of Evans **(4)**, and cannot be treated by his procedure.

As an application of the general method, we derive an asymptotic expansion of the Whittaker function $M_{2a+b,m}(z)$ for numerically large m, $\mathscr{R}(m) > 0$, and a and b bounded.

2. Formal solutions. By a change of notation we write the differential equation as

$$L[u] \equiv \frac{d^2u}{dz^2} - \left\{\lambda^2 \frac{\phi^2(z)}{z^2} + \lambda \frac{\tau(\lambda)}{z^2} + \lambda \frac{\theta(z, \lambda)}{z}\right\} u = 0, \tag{1}$$

with

$$\theta(z, \lambda) = \sum_0^\infty \theta_j(z)/\lambda^j.$$

The functions $\theta_j(z)$ and $\phi(z)$ are analytic in D, and $\phi(z)$ has no zeros there. By the transfer of a constant to λ^2, we adopt the normalization $\phi(0) = 1$.

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[1]The constant N is to be chosen so large that it satisfies all subsequent requirements to be placed upon it.

Thus the function

$$\omega(z) = \frac{\phi(z) - 1}{z}, \quad z \neq 0, \quad \omega(0) = \phi'(0),$$

is analytic. Lastly we define

$$\Omega(z) = \int_0^z \omega(t)\, dt.$$

The exponents of the equation $L[u] = 0$ about $z = 0$ are $\frac{1}{2} \pm \mu(\lambda)$, where

$$\mu(\lambda) = \lambda\left\{1 + \frac{\tau(\lambda)}{\lambda} + \frac{1}{4\lambda^2}\right\}^{\frac{1}{2}}.$$

It is convenient to introduce the notation that

$$\mu(\lambda) - \lambda = \kappa(\lambda), \quad \kappa(\lambda) = \sum_0^\infty \kappa_j/\lambda^j.$$

We now choose n to be a non-negative integer which will remain fixed for the remainder of the discussion. In order to find formal solutions of $L[u] = 0$, let us consider the functions $y_{\pm n}(z, \lambda)$ defined by the relations

$$y_{\pm n}(z, \lambda) = z^{\frac{1}{2} \pm \mu(\lambda)} e^{\pm \lambda \Omega(z)} A_{\pm n}(z, \lambda). \tag{2}$$

Here the functions $A_{\pm n}(z, \lambda)$ are certain polynomials in $1/\lambda$ with coefficients analytic in z, i.e.,

$$A_{\pm n}(z, \lambda) = \sum_0^n \alpha_{\pm j}(z)/\lambda^j.$$

By direct computation

$$L[y_{\pm n}] = z^{-\frac{1}{2} \pm \mu(\lambda)} e^{\pm \lambda \Omega(z)} \frac{H_{\pm n}(z, \lambda)}{\lambda^n}, \tag{3}$$

with

$$\begin{aligned} H_{\pm n}(z, \lambda) = \lambda^n [zA_n''(z, \lambda) + \{1 \pm 2[\lambda\phi(z) + \kappa(\lambda)]\} A_{\pm n}'(z, \lambda) \\ + \lambda\{2\kappa(\lambda)\, \omega(z) - \theta(z, \lambda) \pm \phi'(z)\}\, A_{\pm n}(z, \lambda)]. \end{aligned}$$

The primes denote differentiation with respect to z.

Expanding the functions $H_{\pm n}(z, \lambda)$ in powers of λ and equating the coefficients of $\lambda^{n+1}, \lambda^n, \ldots, \lambda$ to zero, one obtains a recursive set of first order linear differential equations in the functions $\alpha_{\pm j}(z)$. These equations are satisfied, and hence the functions $H_{\pm n}(z, \lambda)$ are bounded provided that

$$\alpha_{\pm 0}(z) = \phi^{-\frac{1}{2}} \exp\left(\mp \int_0^z \{2\kappa_0 \omega(z) - \theta_0(z)\} \frac{dt}{2\phi(t)}\right),$$

$$\alpha_{\pm m}(z) = \alpha_{\pm 0}(z) \int_0^z \frac{F_{\pm m}(t)}{2\phi(t)\, \alpha_{\pm 0}(t)}\, dt, \qquad m = 1, 2, \ldots, n,$$

where

$$\begin{aligned} F_{\pm m}(z) = -2 \sum_{j=0}^{m-1} \kappa_j \alpha_{\pm(m-j-1)}(z) \\ \mp \left\{[z\alpha_{\pm(m-1)}'(z)]' + \sum_{j=1}^{m} [2\kappa_j \omega(z) - \theta_j(z)]\, \alpha_{\pm(m-j)}(z)\right\}. \end{aligned}$$

The coefficients $\alpha_{\pm j}(z)$ so defined are analytic; and $\alpha_{\pm 0}(0) = 1$, $\alpha_{\pm m}(0) = 0$, $m = 1, 2, \ldots, n$. Moreover, $H_{\pm n}(z, \lambda)$ and $A_{\pm n}(z, \lambda)$ are also analytic for $z \in D$ and $|\lambda| > N$.

If in the above discussion the integer n were to be replaced by ∞, one would have the result that $L[y_{\pm\infty}] = 0$. Formally, then, the infinite series would satisfy the given differential equation. Actually, of course, they might be divergent.

It is to be noted that the above procedure is closely analogous to that employed in **(1)** near an ordinary point of the equation. The modifications relate directly to the presence of the pole.

3. A related differential equation. The Wronskian of the functions $y_{+n}(z, \lambda)$ and $y_{-n}(z, \lambda)$,

$$W(z, \lambda) = y_{+n}(z, \lambda)\, y'_{-n}(z, \lambda) - y'_{+n}(z, \lambda)\, y_{-n}(z, \lambda), \tag{4}$$

is, by simple computation,

$$W(z, \lambda) = \begin{vmatrix} A_{+n} & A_{-n} \\ zA'_{+n} + [\tfrac{1}{2} + \lambda\phi + \kappa] A_{+n} & zA'_{-n} + [\tfrac{1}{2} - (\lambda\phi + \kappa)] A_{-n} \end{vmatrix}.$$

From this relation we see that $W(z, \lambda)$ is analytic for $z \in D$ and $|\lambda| > N$ and that its value at $z = 0$ is $-2\mu(\lambda)$. Differentiating both sides of (4) and replacing the functions $y_{\pm n}''(z, \lambda)$ by their equivalents obtained from (3), one finds that

$$W'(z, \lambda) = \frac{1}{\lambda^n}\{A_{+n}H_{-n} - A_{-n}H_{+n}\}.$$

Since $W(0, \lambda) = -2\mu(\lambda)$, by integrating this expression from zero to z, we may conclude that

$$W(z, \lambda) = -2\mu(\lambda) + z\, B(z, \lambda)/\lambda^n. \tag{5}$$

Here $B(z, \lambda)$ designates a function which is analytic and hence bounded for $z \in D$ and $|\lambda| > N$. For future use, we adopt $B(z, \lambda)$ as generic notation for such a function.

It is familiar that the linearly independent functions $y_{+n}(z, \lambda)$ and $y_{-n}(z, \lambda)$ satisfy the second order differential equation

$$\frac{d^2y}{dz^2} - \frac{W'}{W}\frac{dy}{dz} - \left\{\frac{y''_{+n}y'_{-n} - y'_{+n}y''_{-n}}{W}\right\} y = 0. \tag{6}$$

Again evaluating $y_{\pm n}''(z, \lambda)$ by means of (3), the bracketed quantity is found to be

$$\frac{\lambda^2 P(z, \lambda)}{z^2} + \frac{G(z, \lambda)}{z\lambda^n},$$

where

$$G(z, \lambda) = \frac{1}{W}\begin{vmatrix} H_{+n} & H_{-n} \\ zA'_{+n} + [\tfrac{1}{2} + \lambda\phi + \kappa] A_{+n} & zA'_{-n} + [\tfrac{1}{2} - (\lambda\phi + \kappa)] A_{-n} \end{vmatrix}.$$

The transformation of variable

$$v(z, \lambda) = \left[\frac{-2\mu(\lambda)}{W(z, \lambda)}\right]^{\frac{1}{2}} y(z, \lambda) \tag{7}$$

carries the differential equation (6) into

$$\frac{d^2v}{dz^2} - \left\{\frac{\lambda^2 P(z, \lambda)}{z^2} + \frac{B(z, \lambda)}{z\lambda^n}\right\} v = 0. \tag{8}$$

This equation is in an evident sense an nth degree approximation to the given differential equation (1). The solutions $v_{\pm n}(z, \lambda)$ of (8) which correspond to the solutions $y_{\pm n}(z, \lambda)$ of (6) by the transformation (7) are, from (5), of the form

$$v_{\pm n}(z, \lambda) = z^{\frac{1}{2}\pm\mu(\lambda)} e^{\pm\lambda\Omega(z)}\left[\sum_0^n \frac{\alpha_{\pm j}(z)}{\lambda^j} + z\frac{B(z, \lambda)}{\lambda^{n+1}}\right].$$

These functions are linearly independent. Their Wronskian, which is constant as to z, has the value $-2\mu(\lambda)$.

4. Comparison of $v_{+n}(z, \lambda)$ with solutions of $L[u] = 0$. We introduce the function

$$\sigma(z) = ze^{\Omega(z)}.$$

Clearly, $\sigma(z)$ is schlicht **(3)**. Let us confine our attention to a region $E \subset D$ whose one-to-one image, $\mathscr{E}$, in the σ-plane is a circle centered at the origin. For definiteness we restrict the argument of z to be single-valued, making a cut along the negative real axis in $\mathscr{E}$.

By familiar use of the method of variation of parameters, the differential equation $L[u] = 0$ may be shown to be equivalent to an integral equation of the form

$$u(z, \lambda) = v(z, \lambda) + \frac{1}{\lambda^{n+1}}\int_{z_0}^{z} \{v_{+n}(z, \lambda)\, v_{-n}(t, \lambda) - v_{+n}(t, \lambda)\, v_{-n}(z, \lambda)\} \cdot \frac{B(t, \lambda)}{t} u(t, \lambda)\, dt, \tag{9}$$

where $v(z, \lambda)$ is any solution of (8) and z_0 is any point of E (including $z_0 = 0$, provided the integral is convergent).

If the σ-plane is subdivided into four quadrants, it is possible to choose z_0 and $v(z, \lambda)$ in equation (9) so that the integral in this equation may be estimated and asymptotic forms for two linearly independent solutions of (9) derived in each quadrant. For illustrative purposes, a discussion of the derivation of the asymptotic forms of two of these solutions is given below.

(a) *Solutions subdominant*[2] *in* λ *for small* z. We suppose for the moment that $\mathscr{R}(\mu) > 0$. In (9) choose $v(z, \lambda) = v_{+n}(z, \lambda)$, $z_0 = 0$, and the path of integration

[2] In the sense employed, for example, by Langer **(6)**.

the image of a radius vector in $\mathscr{E}$. We call the solution of (9) which is thus determined $u_s(z, \lambda)$. It will be evident that this solution is well defined for all z in E. Introducing the notation

$$V_{\pm s}(z, \lambda) = z^{-\frac{1}{4}+s(\lambda)} e^{\mp\lambda\Omega(z)} v_{\pm s}(z, \lambda),$$

$$U_s(z, \lambda) = z^{-\frac{1}{4}-s(\lambda)} e^{-\lambda\Omega(z)} u_s(z, \lambda),$$

we may rewrite equation (9) as

$$U_s(z, \lambda) = V_{+s}(z, \lambda) + \frac{1}{\lambda^{n+1}} \int_0^z K_s(z, t, \lambda)\, U_s(t, \lambda)\, dt, \tag{10}$$

where

$$K_s(z, t, \lambda) = \left\{ V_{+s}(z, \lambda)\, V_{-s}(t, \lambda) - V_{+s}(t, \lambda)\, V_{-s}(z, \lambda) \left[\frac{\sigma(t)}{\sigma(z)} \right]^{2s} e^{-2s[\Omega(t)-\Omega(z)]} \right\} B(t, \lambda).$$

The functions $V_{\pm s}(z, \lambda)$, $B(z, \lambda)$, and the exponential occurring in $K_s(z, t, \lambda)$ are bounded for $z \in E$ and $|\lambda| > N$. The quantity $[\sigma(t)/\sigma(z)]^{2s}$ is bounded on the contour of integration in (10) since $|\sigma(t)| \leqslant |\sigma(z)|$ on the contour and $\mathscr{R}(\mu) \geqslant 0$. Thus $K_s(z, t, \lambda)$ is bounded on the chosen contour which is of finite length. In fact, there is a uniform bound to the lengths of the contours corresponding to each $z \in E$. It follows that

$$\int_0^z |K_s(z, t, \lambda)| \cdot |dt|$$

is also bounded for $z \in E$. As a consequence:[3]

$$U_s(z, \lambda) = V_{+s}(z, \lambda) + \frac{B(z, \lambda)}{\lambda^{n+1}}.$$

Thus

$$u_s(z, \lambda) = z^{\frac{1}{4}+s(\lambda)} e^{\lambda\Omega(z)} \left\{ \sum_0^n \frac{\alpha_j(z)}{\lambda^j} + \frac{B(z, \lambda)}{\lambda^{n+1}} \right\},$$

for $z \in E$ and $|\lambda| > N$.

An analogous discussion can be given when $\mathscr{R}(\mu) \leqslant 0$. Inasmuch as this is the case, we have established the following theorem.

THEOREM 1. *Over each half plane $\mathscr{R}(\pm\mu) \geqslant 0$, the differential equation $L[u] = 0$ admits of a solution $u_{\pm s}(z, \lambda)$ for $z \in E$ and $|\lambda| > N$, where*

$$u_{\pm s}(z, \lambda) = z^{\frac{1}{4}\pm s(\lambda)} e^{\pm\lambda\Omega(z)} \left\{ \sum_0^n \frac{\alpha_{\pm j}(z)}{\lambda^j} + \frac{B(z, \lambda)}{\lambda^{n+1}} \right\}.$$

The functions $\mu(\lambda)$, $\Omega(z)$, and $\alpha_{\pm j}(z)$ are all defined in §2.

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[3] See Lemma of Langer (**5**, p. 479). The extension to a complex variable involves no difficulty.

(b) *Solutions dominant in* λ *for small* z. We now momentarily suppose that $\mathscr{R}(\mu) > 0$ and $\mathscr{I}(\mu) < 0$. In (9) set $v(z, \lambda) = v_{-n}(z, \lambda)$. The contour of integration is to be the curve whose image in $\mathscr{E}$ begins at the point of the boundary of the circle at which $\arg(\sigma) = \pi$ and is composed of an arc of this circle together with a radial line segment terminating at $\sigma(z)$. As we are considering values of $\mu(\lambda)$ in the fourth quadrant, we call the solution of (9) determined by these conditions $u_4(z, \lambda)$. If we write,

$$U_4(z, \lambda) = z^{-\frac{1}{2}+\mu(\lambda)} e^{\lambda\Omega(z)} u_4(z, \lambda),$$

then equation (9) may be written in the form

$$U_4(z, \lambda) = V_{-n}(z, \lambda) + \lambda^{-n-1} \int_{z_0}^{z} K_4(z, t, \lambda)\, U_4(t, \lambda)\, dt,$$

where

$$K_4(z, t, \lambda) = \Big\{ V_{+n}(z, \lambda)\, V_{-n}(t, \lambda) \Big[\frac{\sigma(t)}{\sigma(z)} \Big]^{-2\mu} e^{2\lambda[\Omega(t)-\Omega(z)]} - V_{+n}(t, \lambda)\, V_{-n}(z, \lambda) \Big\} B(t, \lambda).$$

On the present contour both $|\sigma(z)| \leqslant |\sigma(t)|$ and $\arg\sigma(z) \leqslant \arg\sigma(t)$; hence, $[\sigma(t)/\sigma(z)]^{-2\mu}$ is bounded on the contour. Reasoning as before, we may conclude that

$$U_4(z, \lambda) = V_{-n}(z, \lambda) + \frac{B(z, \lambda)}{\lambda^{n+1}}$$

for all $z \in E$ and $|\lambda| > N$. Thus when $z \neq 0$,

$$u_4(z, \lambda) = z^{\frac{1}{2}-\mu(\lambda)} e^{-\lambda\Omega(z)} \Big\{ \sum_0^n \frac{\alpha_{-j}(z)}{\lambda^j} + \frac{B(z, \lambda)}{\lambda^{n+1}} \Big\}.$$

If $\mathscr{R}(\mu) = 0$, the above relation holds for all $z \in E$.

The results of the preceding analysis are included and extended in the following theorem.

THEOREM 2. *Over each quadrant* $(\nu - 1)\pi/2 \leqslant \arg(\mu) \leqslant \frac{1}{2}\nu\pi$, *the differential equation* $L[u] = 0$ *admits of a solution* $u_\nu(z, \lambda)$ *for all* $z \in E$ *except* $z = 0$ *and for* $|\lambda| > N$, *where*

$$u_\nu(z, \lambda) = z^{\frac{1}{2}-\mu(\lambda)} e^{-\lambda\Omega(z)} \Big\{ \sum_0^n \frac{\alpha_{-j}(z)}{\lambda^j} + \frac{B(z, \lambda)}{\lambda^{n+1}} \Big\}, \qquad \nu = 1 \text{ and } 4,$$

$$u_\nu(z, \lambda) = z^{\frac{1}{2}+\mu(\lambda)} e^{\lambda\Omega(z)} \Big\{ \sum_0^n \frac{\alpha_{+j}(z)}{\lambda^j} + \frac{B(z, \lambda)}{\lambda^{n+1}} \Big\}, \qquad \nu = 2 \text{ and } 3.$$

When $\mathscr{R}(\mu) = 0$, the conclusions of the Theorem may be drawn even when $z = 0$. When $2\mu(\lambda)$ is integral, the logarithmic term possibly occurring in $u_\nu(z, \lambda)$ is of the form $c(\lambda)\, ln\, z\, u_{\pm 2}(z, \lambda)$, according as $\nu = 1, 4$ or $\nu = 2, 3$, and is subsumed in the undetermined function $B(z, \lambda)/\lambda^{n+1}$. The connection between the above results and the Fuchsian theory is manifest.

5. An application. Among the special functions which satisfy differential equations of the type $L[u] = 0$ are the Whittaker functions commonly denoted $M_{k,m}(z)$. They are solutions of the differential equation **(2)**

$$\frac{d^2w}{dz^2} + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2}\right) W = 0, \tag{11}$$

The parameters and variable z are taken to be complex. We consider the function $M_{k,m}(z)$ for $|m|$ and $|k|$ large subject to the restrictions that $|z|$ be bounded, that $k = 2am + b$, where the secondary parameters a and b vary boundedly, and that $\mathscr{R}(m) > 0$. For the structure of $M_{k,m}(z)$ when $|z|$ and $|k|$ are bounded but $|m|$ is large see **(2**, p. 93**)**.

If we make the identification

$$\frac{\lambda^2 P(z, \lambda)}{z^2} = \frac{m^2}{z^2}\left(1 - \frac{2az}{m} + \frac{\frac{1}{4}z^2 - bz - \frac{1}{4}}{m^2}\right),$$

a simple computation yields the results that for equation (11)

$$y_{\pm 1}(z, m) = z^{\frac{1}{2}\pm m} e^{\pm az}\left[1 \pm \frac{z}{2m}\left(\frac{z}{8} - b \pm a - \frac{a^2 z}{2}\right)\right].$$

A calculation of $y_{\pm n}(z, m)$ for an n greater than one is feasible as the quantities $\alpha_{\pm n}(z)$ may be seen to involve only polynomials in z, the secondary parameters a and b, and $e^{\pm az}$. The hypotheses of the theory just presented are easily seen to be satisfied by equation (11) for z in any circle with center at the origin provided that $|m|$ is sufficiently large. Further, the bound implied in $B(z, m)$ will not depend upon a and b if $|a|$ and $|b| < M$.

It remains to identify $M_{k,m}(z)$ as a linear combination of the solutions given by the theorems of the last section. Since $M_{k,m}(z)$ and $u_s(z, m)$ are both subdominant for $\mathscr{R}(m) > 0$ and since

$$\lim_{z\to 0} z^{-\frac{1}{2}-m} M_{k,m}(z) = 1, \quad \lim_{z\to 0} z^{-\frac{1}{2}-m} u_s(z, m) = 1,$$

it follows that

$$M_{k,m}(z) = u_s(z, m).$$

Thus,

THEOREM 3.

$$M_{k,m}(z) = z^{\frac{1}{2}+m} e^{-az}\left[1 + \frac{z}{2m}\left(\frac{z}{8} - b + a - \frac{a^2 z}{2}\right) + \frac{B(z, m)}{m^2}\right],$$

when $\mathscr{R}(m) > 0$, $k = 2am + b$, *and* $a, b,$ *and* z *are bounded.*

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DIFFERENTIABLE POINTS OF ARCS IN CONFORMAL 3-SPACE

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Introduction. This paper is a generalization of the classification of the differentiable points in the conformal plane given in **(1)**. The main tools are the intersection and support properties of all the spheres through a differentiable point of an arc in conformal 3-space.

The discussion is also related to the classification **(2)** of the differentiable points of arcs in projective 4-space, since conformal 3-space can be represented on the surface of a 3-sphere in projective 4-space.

1. Pencils of spheres. In the following discussion, p, t, P, Q, and R will denote points of conformal space, while S and C will denote a sphere and a circle respectively. A sphere S decomposes the space into two open regions, its *interior* $\mathring{S}$ and its *exterior* $\hat{S}$. If P is any fixed point which does not lie on S, the "interior" of S may be defined as the class of all points which do not lie on S and which are not separated from P by S. The exterior of S is then the class of points which are separated from P by S. The sphere through a proper circle C and a point $P \not\subset C$ will be denoted by $S(P; C)$. Much of the following discussion will depend on the use of pencils π of spheres and circles, determined by certain incidence and tangency conditions. A circle (point) which is common to all the spheres (circles) of a pencil is called a *fundamental circle* (*fundamental point*) of the pencil. In the pencil π of spheres through a fundamental circle C, there exists one and only one sphere $S(P; \pi)$ of π through any point P which does not lie on C. Similarly, in the pencil π of the spheres (circles) which touch a given sphere (circle) at a given point Q, there is one and only one sphere $S(P; \pi)$ [circle $C(P; \pi)$] of π which passes through any point $P \neq Q$. The *fundamental point* Q is regarded as a *point-sphere* (*point-circle*) belonging to π.

2. Convergence. A sequence of points $P_1, P_2, \ldots$, is said to be *convergent* to P if to every sphere S with $P \subset \mathring{S}$, there corresponds a positive integer $N = N(S)$, such that $P_\nu \subset \mathring{S}$ if $\nu > N$. The convergence of spheres and circles to a point is defined in a similar fashion.

A sequence of spheres $S_1, S_2, \ldots$, is said to be *convergent* to S if to every pair of points $P \subset \mathring{S}$ and $Q \subset \hat{S}$, there corresponds a positive integer $N = N(P, Q)$, such that $P \subset \mathring{S}_\nu$ and $Q \subset \hat{S}_\nu$ for every $\nu > N$.

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Finally a sequence of circles C_ν is said to be convergent to a circle C, if to every circle D which links with C, there corresponds a positive integer $N = N(D)$ such that C_ν links with D whenever ν is greater than N.

3. Arcs. An arc A is the continuous image of a real interval. If a sequence of points of this parameter interval converges to a point p, the corresponding sequence of image points is defined to be convergent to the image of p. The same small italics $p, t, \ldots$, will be used to denote both the points of the parameter interval and their images on A. The *end-(interior-) points* of A are the images of the end-(interior-) points of the parameter interval. A *neighbourhood* of p on A is the image of a neighbourhood of the parameter p on the parameter interval. If p is an interior point of A, this neighbourhood is decomposed by p into two (open) *one-sided neighbourhoods*. The images of distinct points of the parameter interval are considered to be different points of A, even though they may coincide in space. The notation $Q \neq P$ will indicate that the points Q and P do not coincide.

4. Differentiability. Let p be a fixed point of an arc A, and let t be a variable point of A. If P, Q, and p are mutually distinct points, the unique circle through these points will be denoted by $C(P, Q; \gamma_0)$. The symbol γ_0 itself will denote the family of all circles through p, including the point-circle p.

A is called *once-differentiable* at p if the following condition Γ_1 is satisfied:

Γ_1: If the parameter t is sufficiently close to, but different from, the parameter p, the circle $C(P, t; \gamma_0)$ is uniquely defined, and converges if t tends to p.

Thus the limit circle, which will be denoted by $C(P; \gamma_1)$, is independent of the way t converges to p. The family of all such circles, (i.e., the circles $C(P; \gamma_1)$ for all points $P \neq p$), together with the point circle p, will be denoted by γ_1.

A is called *twice-differentiable* at p if, in addition to the condition Γ_1, the following condition is also satisfied:

Γ_2: If the parameter t is sufficiently close to, but different from, the parameter p, the circle $C(t; \gamma_1)$ is uniquely defined, and converges if t tends to p.

The limit circle of the sequence $C(t; \gamma_1)$ will be denoted by $C(\gamma_2)$, the *osculating circle* of A at p, and, occasionally, also by the symbol γ_2 alone.

5. Structure of the families of circles through p. In this section, relations among the families of circles $\gamma_0, \gamma_1, \gamma_2$ are discussed.

THEOREM 1. *Suppose A satisfies condition Γ_1 at p. Then t does not coincide with p if the parameter t is sufficiently close to, but different from, the parameter p.*

Proof. Let P be any point different from p. By condition Γ_1, $C(P, t; \gamma_0)$ is uniquely defined when the parameter t is close to, but different from, the parameter p. Thus $t \neq p$.

THEOREM 2. *Suppose A satisfies condition Γ_1 at p. Then the angle at p between any two circles of γ_1 is* 0.

Proof. Let P, Q, R_1, R_2 be variable points, and let R_1 and R_2 converge to the same point R. Suppose there is a fixed sphere separating R from both P and Q. Then

$$\lim \angle\, [C(P, R_1, R_2),\quad C(Q, R_1, R_2)] = 0 \tag{1}$$

whether or not the circles themselves converge. In particular, the angle between $C(P; \gamma_1)$ and $C(Q; \gamma_1)$ is equal to 0.

COROLLARY 1. *If $C(P; \gamma_1)$ and $C(Q; \gamma_1)$ have another point in common, they are identical; thus there is one and only one circle of γ_1 through each point $P \neq p$.*

COROLLARY 2. *γ_1 consists of those circles C which meet a given circle of γ_1 at p at the angle* 0. *Thus the circles of γ_1 all touch at p.*

Proof. Let $P \subset C$, $P \neq p$. Suppose C meets some circle of γ_1 at angle 0 at p. Then C and $C(P; \gamma_1)$ also meet at angle 0 at p and have the point P in common. Hence they are identical.

COROLLARY 3. *If Γ_1 holds for a single point $P \neq p$, then it holds for all such points.*

Proof. If $Q \neq p$, by (1)

$$\lim \angle\, [C(Q, t, p),\quad C(P, t, p)] = 0.$$

Thus $C(Q, t; \gamma_0)$ converges to the unique circle through Q which touches $C(P; \gamma_1)$ at p.

THEOREM 3. *Suppose A satisfies the conditions Γ_1 and Γ_2 at p. Then*

$$\gamma_0 \supset \gamma_1 \supset \gamma_2. \tag{2}$$

Proof. It is clear that $\gamma_0 \supset \gamma_1$. If $C(\gamma_2) = p$, it belongs to γ_1 by definition. Suppose $C(\gamma_2) \neq p$. Then $C(\gamma_2)$, being the limit of a sequence of circles $C(t; \gamma_1)$ each of which touches a given circle $C(P; \gamma_1)$ of γ_1, must itself touch $C(P; \gamma_1)$ at p. Thus $C(\gamma_2) \in \gamma_1$ (cf. Theorem 2, Corollary 2.)

COROLLARY. *If $P \subset C(\gamma_2)$, $P \neq p$, then $C(\gamma_2) = C(P; \gamma_1)$.*

The conditions Γ_1 and Γ_2 are independent. For example, suppose a rectangular cartesian coordinate system is introduced and the arc

$$x = t,\ y = t^2,\ z = \begin{cases} (1 - \sqrt{1 - t^2 - t^4}) \sin t^{-1}, & 0 < |t| < \tfrac{1}{2} \\ 0 & , \ t = 0 \end{cases}$$

is considered. Γ_1 is satisfied at $t = 0$, but Γ_2 is not satisfied there.

6. Differentiable points of arcs. In addition to the conditions Γ_1 and Γ_2, three more conditions involving spheres are introduced. Suppose P, Q,

and R are any three fixed points such that P, Q, R, and p do not lie on the same circle. It will be convenient to denote the unique sphere through p and the points P, Q, and R by the symbol $S(P, Q, R; \sigma_0)$. σ_0 will denote the family of all spheres through p, including the point-sphere p.

A is called *thrice-differentiable* at p if the following three conditions are satisfied:

Σ_1: The sphere $S(P, Q, t; \sigma_0)$ is uniquely defined when the parameter t is sufficiently close to, but different from, the parameter p, and converges, if t tends to p, to a limit sphere which will be denoted by $S(P, Q; \sigma_1)$.

Σ_2: The sphere $S(P, t; \sigma_1)$ is uniquely defined when the parameter t is sufficiently close to, but different from, the parameter p, and converges, if t tends to p, to a limit sphere which will be denoted by $S(P; \sigma_2)$.

Σ_3: The sphere $S(t; \sigma_2)$ is uniquely defined when the parameter t is sufficiently close to, but different from, the parameter p, and converges, if t tends to p, to a limit sphere which will be denoted by $S(\sigma_3)$.

The families of all the spheres $S(P, Q; \sigma_1)$, (i.e., the spheres $S(P, Q; \sigma_1)$ for all pairs of points P and Q such that $P \neq Q$ and $P, Q \neq p$), together with the point-sphere p, will be denoted by σ_1. Similarly, the family of all the spheres $S(P; \sigma_2)$ (including the point-sphere p if $C(\gamma_2) = p$) will be denoted by σ_2. The unique *osculating sphere* $S(\sigma_3)$ will occasionally be denoted by σ_3 alone.

The point p is called a *differentiable point* of A if A is thrice-differentiable at p.

7. Structure of the families of spheres through p. Although the conditions Γ_1 and Γ_2 are independent, not all the conditions Γ_1, Γ_2, Σ_1, Σ_2, and Σ_3 are independent. In addition, the families of spheres σ_0, σ_1, σ_2, and σ_3 are closely connected with the families of circles γ_0, γ_1, and γ_2.

THEOREM 4. *Suppose A satisfies condition Σ_1 at p. Let C be any fixed circle. Then $t \not\subset C$ if the parameter t is sufficiently close to, but different from, the parameter p.*

Proof. The assertion is clearly true if $p \not\subset C$. Suppose $p \subset C$, and let P, Q, p be mutually distinct points on C. By condition Σ_1, $S(P, Q, t; \sigma_0)$ is defined when t is sufficiently close to p. Thus $t \not\subset C(P, Q, p) = C$.

The following example shows that Γ_1 does not imply Σ_1 in general (cf., however, Theorem 5). Consider the arc

$$x = t,\ y = \begin{cases} t^2 \cos t^{-1}, & 0 < |t| \leqslant 1 \\ 0, & t = 0 \end{cases},\ z = \begin{cases} t^2 \sin t^{-1}, & 0 < |t| \leqslant 1 \\ 0, & t = 0 \end{cases},$$

in the neighbourhood of $t = 0$. If $P = \infty$, $Q = (1, 0, 0)$, and $p = (0, 0, 0)$, the sphere $S(P, Q, t; \sigma_0)$ does not converge, while, e.g., $C(P, t; \gamma_0)$ converges to the x-axis and by Theorem 2, Corollary 3, Γ_1 is satisfied.

THEOREM 5. *If A satisfies Σ_1 at p, then Γ_1 holds there and*

$$C(Q;\gamma_1) = \bigcap_{P\neq Q,p} S(P, Q;\sigma_1). \tag{3}$$

Conversely, let A satisfy Γ_1 at p. Then Σ_1 holds at p for all pairs P, Q such that $P \not\subset C(Q;\gamma_1)$, and $S(P, Q;\sigma_1) = S[P; C(Q;\gamma_1)]$.

Proof. Suppose Σ_1 holds at p. Theorem 4 implies that $t \neq p$ if t is close to p. Let $Q \neq p$. Since $C(Q, t;\gamma_0) \subset S(P, Q, t;\sigma_0)$, any limit circle of $C(Q, t;\gamma_0)$ lies on $S(P, Q;\sigma_1)$ for every choice of $P \neq Q$, p. Thus if $P_1 \not\subset S(P_2, Q;\sigma_1)$, any limit circle of $C(Q, t;\gamma_0)$ lies on $S(P_1, Q;\sigma_1) \cap S(P_2, Q;\sigma_1)$ and is therefore uniquely determined. Hence $C(Q, t;\gamma_0)$ converges and we have

$$C(Q;\gamma_1) = \bigcap_{P\neq Q,p} S(P, Q;\sigma_1).$$

Conversely, suppose that Γ_1 holds. If $P \not\subset C(Q;\gamma_1)$, then $P \not\subset C(Q, t;\gamma_0)$ when t is sufficiently close to p and

$$S[P; C(Q;\gamma_1)] = \lim_{t\to p} S[P; C(Q, t;\gamma_0)] = \lim_{t\to p} S(P, Q, t;\sigma_0). \tag{4}$$

Thus for all pairs of points P and Q such that $P \not\subset C(Q;\gamma_1)$, $S(P, Q, t;\sigma_0)$ converges, Σ_1 is satisfied, and $S(P,Q;\sigma_1)$ is the sphere through P and $C(Q;\gamma_1)$.

COROLLARY. *There is only one sphere of σ_1 which contains two points not on the same circle of γ_1.*

Remark. Condition Γ_1 is still satisfied when Σ_1 is replaced by a weaker assumption:

Suppose $S'_1 = S(P_1, Q_1, t;\sigma_0) \to S_1$, $S'_2 = S(P_2, Q_2, t;\sigma_0) \to S_2$, and suppose further that $S_1 \cap S_2 = C \neq p$. Then Γ_1 holds at p. For, let $S'_1 \cap S'_2 = C'$. Then $C' \to C$ and $C' \supset p$ and t. As in equation (1), $\lim \angle [C(P_1, t;\gamma_0), C'] = 0$. Thus $C(P_1, t;\gamma_0)$ converges to the unique circle through P_1 which touches C at p. By Theorem 2, Corollary 3, Γ_1 holds at p.

If, however, $S_1 \cap S_2 = p$, Γ_1 need not hold; e.g., take $P_1 = \infty$, $Q_1 = (1, 0, 0)$, $P_2 = (0, 0, 2)$, $Q_2 = (1, 0, 1)$, $p = (0, 0, 0)$, and let A be the arc,

$$x = \begin{cases} t \sin t^{-1}, & 0 < |t| \leqslant 1 \\ 0, & t = 0 \end{cases}, \quad y = t, \quad z = t^2.$$

S'_1 converges to the xy-plane, S'_2 converges to the sphere $x^2 + y^2 + z^2 - 2z = 0$, but Γ_1 does not hold.

THEOREM 6. *Suppose Σ_1 holds at p. Choose $C \in \gamma_1$, $C \neq p$. Then σ_1 is the set of all spheres which touch C at p.*

Proof. Suppose a sphere $S(P, Q;\sigma_1)$ of σ_1 meets C in a point $R \neq p$. If $R \subset C(Q;\gamma_1)$, by Theorem 5 and Theorem 2, Corollary 1,

$$S(P, Q;\sigma_1) \supset C(Q;\gamma_1) = C$$

while if $R \not\subset C(Q; \gamma_1)$,

$$S(P, Q; \sigma_1) = S[R; C(Q; \gamma_1)] = S(R, Q; \sigma_1)$$
$$= S[Q; C(R; \gamma_1)] = S(Q; C) \supset C.$$

Conversely, suppose a sphere S touches C at p. If $S \supset C$, then $S \in \sigma_1$ (Theorem 5). If $S \cap C = p$, choose a point $Q \subset S, Q \neq p$. Let $C_0 = S(Q; C) \cap S$. Then C_0 touches C at p. By Theorem 2, Corollary 2, $C_0 \in \gamma_1$. Since $S \supset C_0$ and $C_0 \in \gamma_1$, it follows from Theorem 5 that $S \in \sigma_1$.

THEOREM 7. *If A satisfies Σ_1 and Σ_2 at p, then Γ_1 and Γ_2 will also hold at p, and equations* (3) *and*

$$C(\gamma_2) = \bigcap_{P \neq p} S(P; \sigma_2) \tag{5}$$

will be satisfied there. Conversely, let A satisfy Γ_1 and Γ_2 at p and let $C(\gamma_2) \neq p$. If $P \not\subset C(\gamma_2)$, then Σ_2 will hold at p for P, and $S(P; \sigma_2)$ will be the sphere through P and $C(\gamma_2)$.

Proof. Suppose Σ_1 and Σ_2 hold at p. In view of Theorem 5, we have only to show that Σ_2 implies Γ_2, and that (5) holds. If t is close to p, (3) implies that $C(t; \gamma_1) \subset S(P, t; \sigma_1)$ for every point $P \neq p$. Hence any limit circle of $C(t; \gamma_1)$ lies on $S(P; \sigma_2)$. Thus if $P_1 \not\subset S(P_2; \sigma_2)$ this limit circle lies on $S(P_1; \sigma_2) \cap S(P_2; \sigma_2)$ and is therefore uniquely determined. Hence $C(t; \gamma_1)$ converges, and

$$C(\gamma_2) = \bigcap_{P \neq p} S(P; \sigma_2).$$

Thus Σ_2 implies Γ_2 and (5) holds.

Conversely, suppose that Γ_1 and Γ_2 hold and $C(\gamma_2) \neq p$. If $P \not\subset C(\gamma_2)$, then $P \not\subset C(t; \gamma_1)$ when t is sufficiently close to p, and by Theorem 5,

$$S[P; C(\gamma_2)] = \lim_{t \to p} S[P; C(t; \gamma_1)] = \lim_{t \to p} S(P, t; \sigma_1).$$

Hence $S(P, t; \sigma_1)$ exists and converges. Thus $S(P; \sigma_2) = S[P; C(\gamma_2)]$.

COROLLARY 1. *If A satisfies Σ_1 (Σ_1 and Σ_2) at p, then A is once-(twice-) differentiable there.*

In particular, this implies

COROLLARY 2. *If p is a differentiable point of A, then Γ_1 and Γ_2 hold there.*

COROLLARY 3. $S(\sigma_2) \supset C(\gamma_2)$.

Proof. By (5),

$$S(t; \sigma_2) \supset \bigcap_{P \neq p} S(P; \sigma_2) = C(\gamma_2).$$

Hence $S(\sigma_2) \supset C(\gamma_2)$.

This implies

COROLLARY 4. *If $S(\sigma_3) = p$, then $C(\gamma_2) = p$.*

COROLLARY 5. *If $C(\gamma_2) \neq p$, σ_2 consists of the spheres through $C(\gamma_2)$.*

The conditions Γ_1 and Γ_2 by themselves do not imply Σ_1 in general, whether or not $C(\gamma_2) = p$. Consider, for example, the arc

$$x = t,\ y = \begin{cases} t^3 \sin t^{-1}, & 0 < |t| \leqslant 1 \\ 0, & t = 0 \end{cases},\ z = \begin{cases} t^3 \cos t^{-1}, & 0 < |t| \leqslant 1 \\ 0, & t = 0 \end{cases}$$

which satisfies Γ_1 and Γ_2 at $t = 0$, $C(\gamma_2)$ being the x-axis. When $P = \infty$, $Q = (1, 0, 0)$, the sphere $S(P, Q, t; \sigma_0)$ is a plane through the x-axis, and this plane does not converge when $t \to 0$. Thus Σ_1 is not satisfied.

Condition Σ_1 is a very strong one for it implies not only Γ_1, but, as the following theorems show, Σ_2 and Γ_2 as well, and even Σ_3 in the case $C(\gamma_2) \neq p$.

THEOREM 8. *Suppose A satisfies Σ_1 at p. Then A also satisfies Σ_2 at p.*

Proof. Let P be any point $\neq p$. Theorem 4 implies that t does not lie on $C(P; \gamma_1)$ if t is close to p. Hence by Theorem 5, $S(P, t; \sigma_1) = S[t; C(P; \gamma_1)]$. Let $Q \subset C(P; \gamma_1)$, $Q \neq P, p$. Then $C(P; \gamma_1) = C(P, Q; \gamma_0)$. Thus $S(P, t; \sigma_1) = S[t; C(P, Q; \gamma_0)] = S(P, Q, t; \sigma_0)$, and Σ_1 now implies that

$$\lim_{t \to p} S(P, t; \sigma_1) = S(P, Q; \sigma_1). \tag{6}$$

Since $S(P; \sigma_2)$ exists for each point $P \neq p$, Σ_2 is satisfied.

COROLLARY 1. *If A satisfies Σ_1 at p, it also satisfies Γ_2 there.*

Proof. By Theorem 7, condition Σ_2 implies Γ_2.

COROLLARY 2. *If A satisfies Σ_1 at p, p is a differentiable point of A if and only if $S(t; \sigma_2)$ converges when t tends to p.*

Relation (6) implies

COROLLARY 3. $S(P; \sigma_2) \in \sigma_1$.

THEOREM 9. *Suppose A satisfies Σ_1 (and hence Σ_2, Γ_1, and Γ_2) at p, and suppose $C(\gamma_2) \neq p$. Then A also satisfies Σ_3 at p.*

Proof. If t is close to but different from p, $S(t; \sigma_2)$ is defined. By Theorem 4, $t \not\subset C(\gamma_2)$, and by Theorem 7, $S(t; \sigma_2) = S[t; C(\gamma_2)]$. Let $P \subset C(\gamma_2)$, $P \neq p$. Then by the corollary of Theorem 3, $C(\gamma_2) = C(P; \gamma_1)$ and hence $S(t; \sigma_2) = S[t; C(P, \gamma_1)] = S(P, t; \sigma_1)$. Σ_2 now implies that

$$\lim_{t \to p} S(t; \sigma_2) = \lim_{t \to p} S(P, t; \sigma_1) = S(P; \sigma_2) \tag{7}$$

Thus $S(t; \sigma_2)$ converges and Σ_3 holds.

COROLLARY. *If A satisfies condition Σ_1 at p and if $C(\gamma_2) \neq p$, then p is a differentiable point of A.*

The following example shows that p need not be a differentiable point of A when Σ_1 is satisfied and $C(\gamma_2) = p$. Consider the arc defined by

$$x = t^2, y = t^3, z = \begin{cases} t^4 \sin t^{-1}, & 0 < |t| \leqslant 1 \\ 0 & , t = 0 \end{cases}.$$

It can readily be verified that A satisfies Σ_1 at $t = 0$, and that the spheres of σ_2 touch the xy-plane at the origin. Thus $C(\gamma_2)$ is a point circle. However, as t tends to 0, $S(t; \sigma_2)$ oscillates and $x^2 + y^2 + z^2 \pm z = 0$ are two accumulation spheres of the sequence $S(t; \sigma_2)$. Thus Σ_2 does not hold at $t = 0$.

THEOREM 10. *Let Σ_1 hold at p, and let $C(\gamma_2) = p$. Then σ_2 is the set of spheres which touch a given sphere of σ_2 at p.*

Proof. Let P and Q be variable points and let C be a variable circle converging to a fixed point. Suppose there is a fixed sphere which separates this point from P and Q. Then

$$\lim \angle\, [S(P; C), S(Q; C)] = 0$$

whether or not the spheres $S(P; C)$ and $S(Q; C)$ themselves converge. In particular, let P and Q be fixed points $\neq p$ and let $C = C(t; \gamma_1) \to p$, as $t \to p$, $t \subset A, t \neq p$. Then

$$\angle\, [S(P; \sigma_2), S(Q; \sigma_2)] = \lim_{t \to p} \angle\, [S(P, t; \sigma_1), S(Q, t; \sigma_1)] = 0. \tag{8}$$

Hence any two spheres of σ_2 touch at p.

Conversely, let S be a sphere which touches $S(P; \sigma_2)$. Choose a point $Q \subset S$, $Q \neq p$. Then $S(Q; \sigma_2)$ also touches $S(P; \sigma_2)$ at p and $S(Q; \sigma_2) = S$. Thus $S \in \sigma_2$.

COROLLARY 1. *σ_2 is the family of spheres, the intersection of any two of which is $C(\gamma_2)$* (cf. Theorem 7, Corollary 5).

COROLLARY 2. *There is one and only one sphere of σ_2 through each point $\not\subset C(\gamma_2)$; that is, if $Q \subset S(P; \sigma_2)$, $Q \not\subset C(\gamma_2)$, then $S(P; \sigma_2) = S(Q; \sigma_2)$.*

THEOREM 11. *If p is a differentiable point of A, then*

$$\sigma_0 \supset \sigma_1 \supset \sigma_2 \supset \sigma_3. \tag{9}$$

Proof. Evidently $\sigma_0 \supset \sigma_1$. Theorem 8, Corollary 3 shows that $\sigma_1 \supset \sigma_2$. This can also be seen as follows:

Let $P \neq p$. By Theorem 6, any sphere $S(P; \sigma_2)$ of σ_2 is the limit of a sequence of spheres $S(P, t; \sigma_1)$ each of which touches a proper circle $C \in \gamma_1$ at p. Thus $S(P; \sigma_2)$ also touches C at p, and $S(P; \sigma_2) \in \sigma_1$.

Let $C(\gamma_2) \neq p$. By Theorem 7, Corollary 5, σ_2 is the set of all the spheres through $C(\gamma_2)$. Hence $S(\sigma_3)$, being the limit of a sequence of spheres through $C(\gamma_2)$, is itself a sphere through $C(\gamma_2)$, and thus a sphere of σ_2. Relation (7) also implies that $\sigma_2 \supset \sigma_3$ when $C(\gamma_2) \neq p$.

Suppose $C(\gamma_2) = p$. By Theorem 10, σ_2 is the set of all the spheres which touch a given sphere $\neq p$ of σ_2 at p. Hence $S(\sigma_2)$, being the limit of a sequence of such tangent spheres, is itself a sphere of σ_2.

This section can be summarized by the following remark:

Let p be a differentiable point of an arc A. Let $P \neq p$. In addition, if $S(\sigma_3) \neq p$, let $P \subset S(\sigma_3)$.

Let

$$C = \begin{cases} C(\gamma_2) \text{ if } C(\gamma_2) \neq p \\ C(P; \gamma_1) \text{ if } C(\gamma_2) = p \end{cases}, \quad S = \begin{cases} S(\sigma_3) \text{ if } S(\sigma_3) \neq p \\ S(P; \sigma_2) \text{ if } S(\sigma_3) = p \end{cases}.$$

Then $C \subset S$, and the structures of γ_1, σ_1, and σ_2 are completely determined by C and S.

8. Support and intersection. Let p be an interior point of an arc A. Then we call p a *point of support* (*intersection*) with respect to a sphere S, if a sufficiently small neighbourhood of p is decomposed by p into two one-sided neighbourhoods which lie in the same region (in different regions) bounded by S. S is then called a *supporting* (*intersecting*) sphere of A at p. Thus S supports A at p if $p \not\subset S$. By definition, the point-sphere p always supports A at p.

It is possible for a sphere to have points $\neq p$ in common with every neighbourhood of p on A (cf., e.g., equation II, §10). In this case S neither supports nor intersects A at p.

9. Intersection and support properties of the families $\sigma_0 - \sigma_1$, $\sigma_1 - \sigma_2$, and $\sigma_2 - \sigma_3$. Throughout the remainder of the paper, the point p is assumed to be a differentiable interior point of A.

THEOREM 12. *Every sphere $\neq S(\sigma_3)$ either supports or intersects A at p.*

Proof. If a sphere S neither supports nor intersects A at p, then $p \subset S$ and there exists a sequence of points $t \to p$, $t \subset A \cap S$, $t \neq p$. We may assume that conditions Σ_1, Σ_2, and Σ_3 hold for this sequence. Choose points P and Q on S such that P, Q, and p are mutually distinct. Then condition Σ_1 implies $S = S(P, Q, t; \sigma_0)$ for each t, and hence $S = S(P, Q; \sigma_1)$.

By Theorem 5, $S = S(P, Q; \sigma_1) \supset C(P; \gamma_1)$. By Theorem 4, $t \not\subset C(P; \gamma_1)$ and again by Theorem 5, $S = S[t; C(P; \gamma_1)] = S(P, t; \sigma_1)$. Condition Σ_2 now implies that $S = S(P; \sigma_2)$.

Finally, by Theorem 7, $S \supset C(\gamma_2)$, and by Theorem 4, $t \not\subset C(\gamma_2)$. If $C(\gamma_2) \neq p$, Theorem 7 implies that $S = S[t; C(\gamma_2)] = S(t; \sigma_2)$, while if $C(\gamma_2) = p$, Theorem 10 implies that $S = S(t; \sigma_2)$. Applying the condition Σ_3, we are led to the conclusion $S = S(\sigma_3)$.

THEOREM 13. *If $S(\sigma_3) = p$, then the spheres of $\sigma_2 - \sigma_3$ all intersect A at p, or they all support.*

Proof. Let S' and S'' be two distinct spheres of $\sigma_2 - \sigma_3$. Since $S(\sigma_3) = p$, Theorem 7, Corollary 4 implies that $C(\gamma_2) = p$, and Theorem 10 implies

that S' and S'' touch at p. Thus we may assume that $S'' \subset (p \cup \dot{S}')$ and $S' \subset (p \cup \dot{S}'')$. Suppose now, for example, that S' supports A at p while S'' intersects. Then $A \cap \dot{S}''$ is not void and hence we may assume $A \subset (p \cup \dot{S}')$. Let $t \to p$ in $A \cap \dot{S}''$; thus $t \subset \dot{S}'' \cap \dot{S}'$. Hence $S(t; \sigma_2) \subset (\dot{S}'' \cap \dot{S}') \cup p$. Consequently $S(t; \sigma_2)$ cannot converge to $S(\sigma_2) = p$ as $t \to p$. Thus S' and S'' must both support or both intersect A at p.

THEOREM 14. *If $S(\sigma_2) \neq p$ and $C(\gamma_2) = p$, then every sphere of $\sigma_2 - \sigma_2$ supports A at p.*

Proof. Suppose $C(\gamma_2) = p$, so that the spheres of σ_2 all touch at p (Theorem 10). Let $S \in \sigma_2$, $S \neq S(\sigma_2)$, $S \neq p$. If a sequence of points t exists such that $t \subset A \cap \dot{S}$, $t \to p$, then each $S(t; \sigma_2)$ lies in the closure of $\dot{S}$. Hence $S(\sigma_2)$ will lie in the same domain, and therefore even in $p \cup \dot{S}$. Similarly, the existence of a sequence $t' \subset \dot{S} \cap A$, $t' \to p$, implies that $S(\sigma_2) \subset p \cup \dot{S}$. Thus if S intersects A at p, $S(\sigma_2) \subset (p \cup \dot{S}) \cap (p \cup \dot{S}) = p$; in other words, $S(\sigma_2) = p$.

THEOREM 15. *All the spheres of $\sigma_0 - \sigma_1(\sigma_1 - \sigma_2; \sigma_2 - \sigma_2)$ support A at p, or they all intersect.*

Proof. Let S' and S'' be two distinct spheres of $\sigma_0 - \sigma_1$ $(\sigma_1 - \sigma_2; \sigma_2 - \sigma_2)$. Suppose, for the moment, that the intersection $S' \cap S''$ is a proper circle $C_0 = C(P, Q; \gamma_0)$ $[C_1 = C(P; \gamma_1)$; $C_2 = C(\gamma_2)]$ (cf. equations (3) and (5)). Suppose, for example, that S' intersects while S'' supports A at p. With no loss in generality, we may assume that $A \subset \dot{S}'' \cup p$. Thus $A \cap \dot{S}'$ and $A \cap \dot{S}'$ are not void. If $t \subset A \cap \dot{S}'$, by Theorems 4, 5, and 7, $S(P, Q, t; \sigma_0) = S(t; C_0)$ $[S(P, t; \sigma_1) = S(t; C_1); S(t; \sigma_2) = S(t; C_2)]$ lies in the closure of

$$(\dot{S}' \cap \dot{S}'') \cup (\dot{S}' \cap \dot{S}'').$$

Letting $t \to p$ on A, we conclude that $S(P, Q; \sigma_1)$ $[S(P; \sigma_2); S(\sigma_2)]$ lies in the same closed domain. By letting t' converge to p through $\dot{S}' \cap A$, we obtain symmetrically that $S(P, Q; \sigma_1)$ $[S(P; \sigma_2); S(\sigma_2)]$ also lies in the closure of

$$(\dot{S}' \cap \dot{S}'') \cup (\dot{S}' \cap \dot{S}'').$$

Hence $S(P, Q; \sigma_1)$ $[S(P; \sigma_2); S(\sigma_2)]$ lies in the intersection $S' \cup S''$ of these two domains, i.e., $S(P, Q; \sigma_1)$ $[S(P; \sigma_2); S(\sigma_2)]$ is either S' or S'', contrary to our assumptions. Thus S' and S'' both support or they both intersect in this case.

Suppose now that $S' \cap S'' = p$. In view of Theorems 13 and 14 there remain to be considered only the cases where S' and S'' belong to the family $\sigma_0 - \sigma_1$, or both belong to $\sigma_1 - \sigma_2$.

By Theorem 6, any sphere S through p which does not touch a circle C of γ_1 belongs to $\sigma_0 - \sigma_1$; by Theorem 6, Theorem 7, Corollary 5, and Theorem 10, any sphere S which touches a circle C of γ_1, but does not contain $C(\gamma_2)$ in case $C(\gamma_2) \neq p$, and does not touch a sphere of σ_2 in case $C(\gamma_2) = p$, belongs to $\sigma_1 - \sigma_2$. Hence there exists a sphere S of $\sigma_0 - \sigma_1$ $(\sigma_1 - \sigma_2)$ which intersects S' and S'' respectively in a proper circle. From the

above, S and S', and also S and S'', both support or both intersect A at p. Thus S' and S'' both support or both intersect A at p.

THEOREM 16. *If $C(\gamma_2) \neq p$, every sphere of $\sigma_1 - \sigma_2$ supports A at p.*

Proof. Suppose $S \in \sigma_1 - \sigma_2$ intersects A at p. Let $t \to p$, $t \subset A \cap \dot{S}$, $t \neq p$. By Theorem 6, $C(t; \gamma_1)$ touches S at p and hence $C(t; \gamma_1) \subset \dot{S} \cup p$. Since $C(t; \gamma_1) \to C(\gamma_2)$ it follows that $C(\gamma_2) \subset \dot{S} \cup S$. If t' converges to p through $A \cap \dot{S}$, it follows symmetrically that $C(\gamma_2) \subset \dot{S} \cup S$. Thus $C(\gamma_2) \subset S$. Since $S \notin \sigma_2$, however, Theorem 7, Corollary 5 implies that $C(\gamma_2) = p$.

10. Characteristics and a classification of the differentiable points. The *characteristic* $(a_0, a_1, a_2, a_3; i)$ of a differentiable point p of an arc A is defined as follows:

$a_0, a_1, a_2 = 1$ or 2; $a_3 = 1, 2$, or ∞; $i = 1, 2$, or 3.

a_0 is even or odd according as the spheres of $\sigma_0 - \sigma_1$ support or intersect.

$a_0 + a_1$ is even or odd according as the spheres of $\sigma_1 - \sigma_2$ support or intersect.

$a_0 + a_1 + a_2$ is even or odd according as the spheres of $\sigma_2 - \sigma_3$ support or intersect.

$a_0 + a_1 + a_2 + a_3$ is even if $S(\sigma_3)$ supports and odd if $S(\sigma_3)$ intersects, while $a_3 = \infty$ if $S(\sigma_3)$ neither supports nor intersects.

$i = 1$ if $C(\gamma_2) \neq p$; $i = 2$ if $C(\gamma_2) = p$ and $S(\sigma_3) \neq p$; and $i = 3$ if $S(\sigma_3) = p$.

Theorems 16, 14, and the convention that $S(\sigma_3)$ supports when it is the point-sphere, lead to the restriction on the characteristic $(a_0, a_1, a_2, a_3; i)$ that $a_0 + \ldots + a_i$ is even. As a result, there are just 32 types of differentiable points; 12 when $i = 1$, 12 when $i = 2$, and 8 when $i = 3$.

Examples of each of the 32 types are given by the curves

$$\text{(I)} \qquad x = t^m, \quad y = t^n, \quad z = t^r,$$

for the cases $a_3 = 1$ or 2, and

$$\text{(II)} \qquad x = t^m, y = t^n, z = \begin{cases} t^r \sin t^{-1}, & \text{if } 0 < |t| < 1 \\ 0 & , t = 0 \end{cases},$$

for the cases $a_3 = \infty$, all relative to the point $t = 0$. The indices, m, n, and r are positive integers and $m < n < r$. The different types are determined by the parities of the indices m, n, and r and the relative magnitudes of m, n, r, and $2m$. In each of these examples the circles of γ_1 and the spheres of σ_1 touch the x-axis at the origin. In the case $i = 1$, σ_2 is the family of planes through the x-axis; while in each of the cases $i = 2$ or 3, σ_2 is the family of spheres which touch the xy-plane at the origin (cf. Remark at the end of §7).

The first of the following tables lists examples of all the types of differentiable points together with their characteristics, while the second table summarizes the properties of these types. Congruences are mod 2.

TABLE I

$(a_0, a_1, a_2, a_3; i)$

Equation	$i = 1$	$m < 2m < n < r$			$i = 2$	$m < n < 2m < r$			$i = 3$	$m < n < r < 2m$		
I	(1, 1, 1, 1; 1)	$m = 1$	$n = 1$	$r = 0$	(1, 1, 2, 1; 2)	$m = 1$	$n = 0$	$r = 1$	(1, 1, 1, 1; 3)	$m = 1$	$n = 0$	$r = 1$
	(1, 1, 1, 2; 1)			$r = 1$	(1, 1, 2, 2; 2)			$r = 0$	(1, 1, 2, 2; 3)			$r = 0$
	(1, 1, 2, 1; 1)		$n = 0$	$r = 1$	(1, 2, 1, 1; 2)		$n = 1$	$r = 1$	(1, 2, 2, 1; 3)		$n = 1$	$r = 1$
	(1, 1, 2, 2; 1)			$r = 0$	(1, 2, 1, 2; 2)			$r = 0$	(1, 2, 1, 2; 3)			$r = 0$
	(2, 2, 1, 1; 1)	$m = 0$	$n = 1$	$r = 0$	(2, 1, 1, 1; 2)	$m = 0$	$n = 1$	$r = 1$	(2, 1, 1, 2; 3)	$m = 0$	$n = 1$	$r = 0$
	(2, 2, 1, 2; 1)			$r = 1$	(2, 1, 1, 2; 2)			$r = 0$	(2, 1, 2, 1; 3)			$r = 1$
	(2, 2, 2, 1; 1)		$n = 0$	$r = 1$	(2, 2, 2, 1; 2)		$n = 0$	$r = 1$	(2, 2, 1, 1; 3)		$n = 0$	$r = 1$
	(2, 2, 2, 2; 1)			$r = 0$	(2, 2, 2, 2; 2)			$r = 0$	(2, 2, 2, 2; 3)			$r = 0$
II	(1, 1, 1, ∞; 1)	$m = 1$	$n = 1$		(1, 1, 2, ∞; 2)	$m = 1$	$n = 0$					
	(1, 1, 2, ∞; 1)		$n = 0$		(1, 2, 1, ∞; 2)		$n = 1$					
	(2, 2, 1, ∞; 1)	$m = 0$	$n = 1$		(2, 1, 1, ∞; 2)	$m = 0$	$n = 1$					
	(2, 2, 2, ∞; 1)		$n = 0$		(2, 2, 2, ∞; 2)		$n = 0$					

TABLE II

i	$C(\gamma_2)$	$S(\sigma_3)$	Characteristic $(a_0, a_1, a_2, a_3; i)$	Restrictions		Examples: (I) or (II)				No. of types
								$C(\gamma_2)$	$S(\sigma_3)$	
1	$\neq p$	$\neq p$	$(a_0, a_1, a_2, a_3; 1)$ $a_3 = 1$ or 2 $(a_0, a_1, a_2, \infty; 1)$	$\sigma_1 - \sigma_2$ supports	$a_0 + a_1 \equiv 0$	$m < 2m < n < r$	I II	x-axis	xy-plane	8 4
2	$=p$	$\neq p$	$(a_0, a_1, a_2, a_3; 2)$ $a_3 = 1$ or 2 $(a_0, a_1, a_2, \infty; 2)$	$\sigma_2 - \sigma_3$ supports	$a_0 + a_1 + a_2 \equiv 0$	$m < n < 2m < r$	I II	$x = y = z = 0$	xy-plane	8 4
3	$=p$	$=p$	$(a_0, a_1, a_2, a_3; 3)$ $a_3 = 1$ or 2	σ_3 supports	$a_0 + a_1 + a_2 + a_3 \equiv 0$	$m < n < r < 2m$	I	$x = y = z = 0$	$x = y = z = 0$	8

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PENCILS OF POLARITIES IN PROJECTIVE SPACE

SEYMOUR SCHUSTER

1. Introduction. A polarity in complex projective space of two dimensions (S_2) is completely determined by a self-polar triangle ABC, and a pair of corresponding elements: a point P and its polar line p. We denote the polarity by $(ABC)(Pp)$. We follow Coxeter **(2)** in defining a *pencil of polarities* as the ∞^1 polarities $(ABC)(Pp)$ where A, B, C, P are fixed while p varies in a pencil of lines. Coxeter has developed this notion and has shown that two of the classical types of pencils of conics are, in fact, the conics which arise from such pencils of polarities. The pencils of conics so derived are the general system with four distinct points in common and the double-contact system where the conics have in common two distinct points and common tangents at these points.

It is well known that a polarity in S_2 is also completely determined by a self-polar pentagon **(2**, p. 64**)**. Suppose we start afresh and define a pencil of polarities by means of such a pentagon. Will this yield anything more general than the previous definition? The answer to this question is in the affirmative, as we shall see in §2.

Sections 3 and 4 deal with a development of pencils of polarities in complex projective space of three dimensions (S_3) by utilizing a self-polar tetrahedron in the definition. Section 5 continues the development in S_3, but by redefining a pencil of polarities, this time using a self-polar skew hexagon. Section 6 will deal with the problem in projective space of n dimensions (S_n).

2. Pentagonal pencils of polarities.

DEFINITION. A *pentagonal pencil of polarities* is the set of polarities determined by the self-polar pentagon $PQRST$, where P, Q, and T remain fixed and the line RS varies in a pencil of lines with R on the fixed line t through Q, and S on the fixed line q through T (Fig. 1).

Each polarity for such a pentagon has

$$P \leftrightarrow p = RS,\ Q \leftrightarrow q = ST,\ R \leftrightarrow r = TP,\ S \leftrightarrow s = PQ,\ T \leftrightarrow t = QR.$$

If we consider a general pentagon (with none of the vertices on its own polar), we can find a fixed self-polar triangle for the pencil. Let A denote the point of intersection $q \cdot t$. Then $A \leftrightarrow a = QT$, and AQT is a fixed self-polar triangle for the entire pencil. If the pencil of lines p varies about a fixed point on a side of AQT, we can arrive at a self-dual system **(2**, 5.82**)**, as well as the more general type of pencil when p varies about a point in general position.

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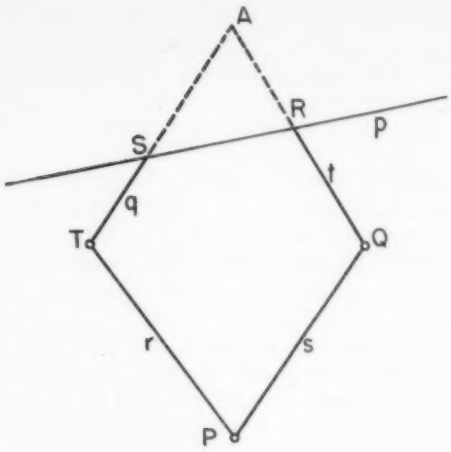

FIGURE 1

Hence, a *pentagonal pencil* yields everything which can be found by a development with a fixed self-polar triangle.

We are particularly interested in finding pentagonal pencils of polarities which have no fixed self-polar triangle. Recalling the construction for a self-polar triangle in a polarity determined by a self-polar pentagon, we see that two alternate vertices of the pentagon must be joined. Now, if there is a pencil of polarities determined by pentagon $PQRST$ with P, Q, T fixed and the line RS varying in a pencil, the only possible fixed self-polar triangle is $QT(QR \cdot ST)$; for, no pair of alternate vertices other than Q and T yield a fixed line. If QT is distinct from $ST = q$, such a triangle is non-degenerate. Hence, attention will be confined to pentagons where QT coincides with ST. That is, *Q is always on its own polar q.*

We can immediately state two theorems of **(2)** which carry over to the case where Q is always a self-conjugate point. Their proofs, which we omit, are very much the same as those supplied in **(2)**, except that there are several additional special cases to consider.

THEOREM 2.1. *The polars of any fixed point X, with respect to a pencil of polarities, form a pencil of lines through a point X'.*

THEOREM 2.2. *The locus of X' as X varies along a fixed line o is a conic or a line. The locus of poles of the fixed line o, with respect to a pencil of polarities is the same conic or line.*

Suppose, for instance, that P', the vertex of the pencil of lines p, is located on r. Then all the polarities in the pencil induce on r the same involution, namely $(PP')(TT_1)$, where T_1 is $t \cdot r$. The invariant points, A and B, of this involution, must be common points of the conics determined by the polarities.

Thus we have a pencil of conics with a common tangent q at Q and two further common points, A and B.

It might be best to introduce some general notation at this point. We will discuss pencils of polarities in i-dimensional space, determined by a configuration generally associated with j points. We denote such a pencil of polarities by the symbol $P^i{}_{jk}$, where k will be an integer indicating a special case of such a pencil. The cases covered by Coxeter were 2-dimensional, defined by triangles; so we call them the $P^2{}_{21}$ and $P^2{}_{22}$ systems. The $P^2{}_{21}$ system is the one yielding a general pencil of conics with four distinct points in common. The $P^2{}_{22}$ system is Coxeter's self-dual system yielding a double-contact pencil of conics. Our pentagonal pencils have yielded $P^2{}_{21}$ and $P^2{}_{22}$, so we will not trouble to give them alternate names $P^2{}_{5k}$. Instead, we begin naming pentagonal systems with the pencil discussed above, which yielded a new pencil of conics. We call this a $P^2{}_{51}$ system.

Suppose we turn our problem around and ask how to determine the pentagon for the pencil of polarities when we are given the pencil of conics with a common tangent q at Q, and two further points A and B in common. The following procedure yields the pentagon. Let $AB = r$ and $T = r \cdot q$ (Fig. 2). Find T_1 on r such that $H(TT_1, AB)$. Let $t = QT_1$. Line t is the polar of T with respect to the pencil of conics.

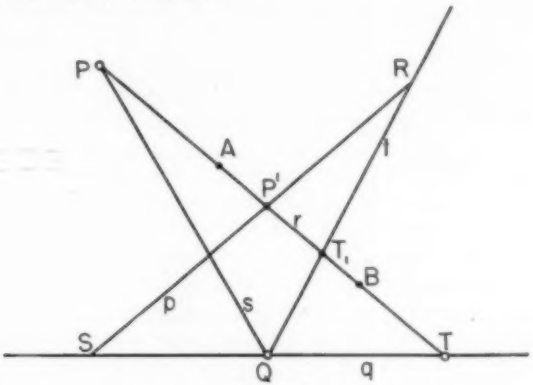

FIGURE 2

Pick a point P arbitrarily on r. Let $s = PQ$. Now two cases arise:

(a) $P \neq A,B$. Find P' on r such that $H(PP', AB)$. An arbitrary line through P' will serve to represent the variable line p. Thus we have a self-polar pentagon $PQRST$, where $p \cdot t = R$ and $p \cdot q = S$, which determines the pencil of conics provided p varies in the pencil about P'.

(b) $P = A$. The point P is self-conjugate. Construct an arbitrary line through P to serve as a representative for the variable pencil. Each arbitrary p is a tangent at P for one conic of the pencil. This time P plays the role of P' so

we have a pentagon which defines the pencil of polarities associated with the conics provided line p varies in the pencil through P.

Suppose we allow the pencil of lines p to have its vertex on q, so that $P' = S$. We will show that such a pencil of polarities leads to a pencil of conics with four-point contact (third order) at point Q. We shall call these polarities a P^2_{52} system. In order to prove our assertion we show that every point on line q possesses the same polar for every polarity in the pencil **(10**, p. 136**)**. From the nature of the pentagon defining the pencil, it is seen that Q, S, and T have fixed polars for the pencil. For a consideration of the general point X on q, we look back at the construction of its polar line x (see Fig. 3). It is clear that x depends not at all upon the position of p, except for the fact that p meets q in S. But this is so for every polarity in the pencil. Hence, point X has a fixed polar line for the entire pencil.

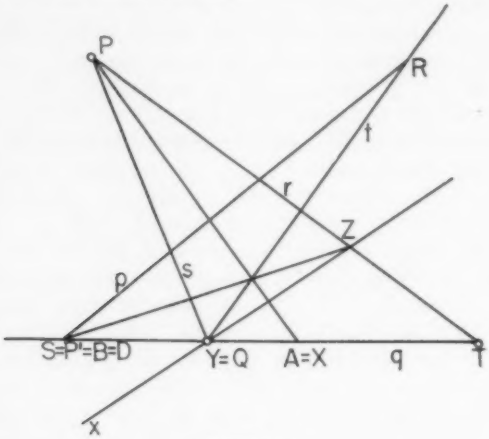

FIGURE 3

N.B. The only way such a pencil of conics can be derived from a pentagonal pencil of polarities is by having point S fixed. For, otherwise, there would be points on q which would not have the same polar for all polarities in the pencil, thus contradicting the definition **(10**, p. 136**)**.

The fact that a four-point contact system of conics is reached by having p rotate about a fixed S can be seen in a simpler and perhaps more elegant manner as follows:

$X \overline{\wedge} x$ is a projectivity of points X on q with their polars through Q. Suppose point X had two distinct polars for two different polarities in the pencil. Then $QSTX \overline{\wedge} qstx_1$ and $QSTX \overline{\wedge} qstx_2$. But this contradicts the fundamental theorem of projective geometry. Therefore $x_1 = x_2$.

Actually, the P^2_{51} system may be reached with fewer restrictions than we have placed on a pentagonal pencil of polarities. That is, line p may vary

about point P', where P' is located anywhere except upon line q. We have already discussed the case where P' is on r. We will now discuss the case where P' is in a general position; the reader might supply the details of the development in the cases where P' is on line t, line s, or even when $P' = P$. The same system of conics is reached in all.

P' is in general position. Since Q is on q, every polarity in the pencil yields a conic with q as the tangent at point Q. The pencil of conics cannot have three-point contact (second order) at Q because point T on q has the same polar for all the polarities, thus contradicting the necessary conditions for three-point contact **(10**, p. 134**)**. The conics cannot have four-point contact, either, owing to the observation on p. 122. Therefore, the pencil must have two-point contact at Q, and two other points A and B in common. Since T and its polar t are fixed for the pencil, A and B must be collinear with T such that $\mathrm{H}(AB, TT_1)$, where $T_1 = t \cdot AT$.

Examination of the special cases where P' is on s, on t or $P' = P$, leads to an actual determination of the points A and B, whereas our examination of the more general case merely established the existence of A and B.

Concerning pencils of conics with three-point contact at Q, we might say that all cases of pentagonal pencils have been investigated and no such systems of conics result; hence, they cannot be derived from a pentagonal pencil of polarities. However, there are more direct methods which are far more convincing:

(1) A pencil with three-point contact (second order) has Q as the only point on the tangent q which possesses the same polar for all polarities in the pencil **(10**, p. 134**)**. But, this is always contradicted by point T, whose polar line is fixed for the pencil.

(2) A pencil of conics with three-point contact at Q has another point, say A, in common. Suppose that such a system of conics arises from a pentagonal pencil. Let $T_1 = t \cdot AT$. The point B, the harmonic conjugate of A with respect to T and T_1, is also common to all the conics. This contradicts the hypothesis that a three-point contact system arises from a pentagonal pencil.

SUMMARY. *The investigation of pentagonal pencils of polarities has shown them to be more general than the triangular pencils. The triangular pencils* P^2_{21} *and* P^2_{22} *can both be derived as pentagonal pencils. Further, two new systems,* P^2_{51} *and* P^2_{52}*, were found to yield pencils of conics:* (1) *with two-point contact at a point and two other distinct points in common, and* (2) *the four-point contact system. It was also shown that the remaining pencil of conics, with three-point contact, cannot be derived from pentagonal pencils.*

3. Tetrahedral pencils of polarities in S_3.

DEFINITION. A *tetrahedral pencil of polarities* is the set of polarities $(ABCD)$ $(P\pi)$, where A, B, C, D, and P are all fixed while plane π varies in an axial pencil.

THEOREM 3.1. *The polar planes of any fixed point X, with respect to a pencil of polarities, form a pencil of planes.*

Proof. We use the following construction **(4)** for the polar plane of an arbitrary point. Let the polarity be $(ABCD)$ $(P\pi)$, and X a point in general position. We seek χ, the polar plane of X. We use the notation

$$ABC = \delta,\ BCD = \alpha,\ ACD = \beta,\ ABD = \gamma,$$

$DP \cdot \delta = P_\delta$, $DX \cdot \delta = X_\delta$, $\pi \cdot \delta = p_\delta$ (Fig. 4). Since DP is the polar line of p_δ, the polarity induced in plane δ is $(ABC)(P_\delta p_\delta)$. By Coxeter's construction **(2**, 5.64**)**, the polar of X_δ in plane δ is line

$$x_\delta = [AP_\delta \cdot (a \cdot P_\delta X_\delta)(p_\delta \cdot AX_\delta)] \cdot [BP_\delta \cdot (b \cdot P_\delta X_\delta)(p_\delta \cdot BX_\delta)].$$

By performing a similar construction in plane α, we can get the polar x_α of the point X_α. The plane determined by $x_\alpha x_\delta$ is the polar plane of point X. In order to prove this it is sufficient to remark that X is conjugate to every point on x_α and to every point on x_δ.

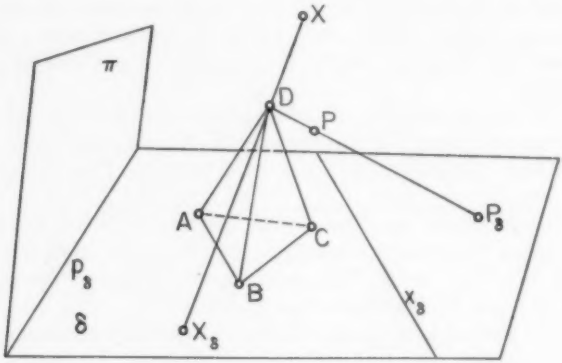

FIGURE 4

Referring to the construction, we proceed with the proof. Let π vary in a pencil about line p'. This line cannot lie in more than two of the four faces. Let α and δ be two in which it does not lie. Then $p' \neq p_\delta$. Line p_δ varies in a pencil (in plane δ) about the point P'_δ. By Coxeter's result **(2**, 5.81**)**, we know that line x_δ likewise varies in a pencil (in plane δ) about a point X'_δ. Similarly, line x_α varies in a pencil (in plane α) about a point X'_α. In each polarity of the pencil we have χ, the polar plane of X, determined by $x_\alpha x_\delta$. χ must pass through X'_α and X'_δ in each of the polarities. Hence, the polar planes of X vary in a pencil about $X'_\alpha X'_\delta$.

In case $X'_\alpha = X'_\delta$, we take X'_β and X'_γ. All four cannot coincide.

Three special cases of tetrahedral pencils are of interest because they lead to systems, highly analogous to the self-dual systems discussed in **(2)**. These arise when line p' meets one or more edges of $ABCD$.

There will be four tetrahedral pencils P^3_{4k}, where k is 0, 1, 2, or 3, according as line p' meets 0, 1, 2 or 3 edges of $ABCD$. It is clear that if p' were to meet more than three edges of the self-polar tetrahedron we would not have a well-defined pencil.

The P^3_{41} system. Suppose p' meets only one edge, say AB, of the self-polar tetrahedron. Then $P'_\delta = P'_\gamma$ is on AB. From the discussion in **(2)** we know that the polarities induced in the two faces through AB are two-dimensional self-dual systems P^2_{32}. Any pair of the involution on AB, together with C and D, will give a fixed self-polar tetrahedron for the entire pencil. Therefore, *we have not one, but ∞^1 tetrahedra which are self-polar for all polarities in the pencil.*

THEOREM 3.2. *If two polarities belong to a P^3_{41} system, their product is a general axial homography* **(1**, p. 385; **5**, pp. 180–183**)**.

Proof. Let p' meet AB in P'_δ. A self-dual system is induced in plane ABC. Therefore, the product of two polarities in P^3_{41} leaves invariant every point on AB. Also, points C and D are invariant, so we must have the pencil of planes with axis CD as an invariant pencil. Thus CD is the tangential-axis and AB is the point-axis of a general axial homography; C and D are the invariant points.

The P^3_{42} system. We suppose now that p' meets exactly two edges, say AB and CD, of the self-polar tetrahedron. This implies that pencils of polarities induced in the faces of $ABCD$ are all self-dual, with the same involutions of points on AB and CD for all the polarities in the pencil. Therefore, if we take any pair of the involution on AB together with any pair of the involution on CD, we get four points which determine a self-polar tetrahedron for the entire pencil. Hence, *∞^2 self-polar tetrahedra are available in a P^3_{42} system.*

THEOREM 3.3. *If two polarities belong to a P^3_{42} system, their product is a hyperbolic biaxial homography.*

Proof. Lines AB and CD are pointwise invariant under the product. Since a hyperbolic biaxial homography is characterized by two such lines, the theorem is proved.

The P^3_{43} system. Suppose p' meets three edges of $ABCD$. It must lie in a face, say plane δ. The polarity induced in δ is, therefore, the same for each member of the pencil. The polarity is always $(ABC)(P_\delta p')$. Hence, the polar planes of X (for all members of the pencil) pass through line x_δ. The involutions of conjugate points induced on AB, BC, and AC are the same for all members of the pencil. By taking any pair of the involution on one of these lines together with the opposite vertex (the pole of the line in plane δ), we can arrive at an infinite number of self-polar triangles. In fact, there are ∞^2 such. Therefore,

we have ∞^3 *tetrahedra* (*formed by D and the* ∞^2 *self-polar triangles in plane* δ) *which are self-polar for all the polarities.* Hence

THEOREM 3.4. *A* P^3_{43} *system admits a plane* δ *upon which the induced polarity is the same for all members of the pencil. The polars of any point P pass through a fixed line* p' *on* δ, *and the poles of any plane* π *form a range of points on a fixed line* DP_δ, *where D is the pole of* δ *for all the polarities.*

(If we define a *range* of polarities by fixing tetrahedron $ABCD$ and plane π, while P, the pole of π, varies in a range, then the above theorem indicates that a P^3_{43} system is also a range of polarities. Therefore, we might call it a self-dual system in S_3.)

THEOREM 3.5. *If two polarities belong to the same* P^3_{43} *system, their product is a spatial homology.*

Proof. In a P^3_{43} system every line through one vertex, say D, of the self-polar tetrahedron is invariant, and every pencil of planes whose axis lies in δ is invariant. But we must show that every point of a plane is invariant and every plane through a point is invariant **(10**, p. 75; **5**, p. 179**)**. Consider any point X in δ. Its polar plane χ meets δ in line x'. If we operate with another member of the self-dual system, we find that $x' \to X$ (because the polarity induced in δ is the same for all members of the system). Hence, $X \to X$, which shows that every point of δ is invariant. Now consider an arbitrary plane ρ through D. Let $\rho \cdot \delta = r'$ and let R' be the pole of r' in δ. Then R, the pole of ρ, is on line DR'. Then operate on R with another member of the pencil. The construction tells us that R maps into a plane through r'. Since every line through D is invariant, the plane $r'D$ is invariant. Hence we have a spatial homology with D as its center and δ its plane of perspectivity.

THEOREM 3.6. *Every spatial homology can be expressed as the product of two polarities belonging to a self-dual system* (P^3_{43}).

Proof. The homology whose plane of perspectivity is δ and whose center is D, with a pair of related points P_1 and P_2 (on a line through D) **(10**, p. 76; **5**, p. 179**)**, is the product of polarities $(ABCD)(P_1\pi)$ and $(ABCD)$ $(P_2\pi)$, where π is an arbitrary plane.

THEOREM 3.7. *The locus of poles of a fixed plane with respect to a pencil of polarities is:*

(a) *a twisted cubic passing through the vertices of the fixed self-polar tetrahedron in a* P^3_{40} *pencil.*

(b) *a conic passing through three vertices of one of the fixed self-polar tetrahedra in a* P^3_{41} *pencil.*

(c) *a line passing through two of the vertices of one of the fixed self-polar tetrahedra in a* P^3_{42} *pencil.*

(d) *a line passing through the vertex common to all the self-polar tetrahedra in a* P^3_{43} *pencil.*

Proof.

(d) For a self-dual system (P^3_{43}), Theorem 3.4 indicates that the locus is such a line.

(a) We consider the general case P^3_{40} with the pencil defined by $(ABCD)(P\pi)$. Let χ be a fixed plane. Take three points Q_1, Q_2, and Q_3 on χ. The intersection of the polar planes of these three points is the pole X of χ. Theorem 3.1 indicates that as π varies in the pencil the polar planes of Q_1, Q_2, Q_3 vary in projective axial pencils. The locus of poles is, therefore, a twisted cubic.

Consider this situation from another point of view (Fig. 5). Examining what happens in plane δ, we see that line x_δ is fixed. The locus of poles of x_δ in δ is a conic through A, B, and C (**2**, 6.81). This conic is the projection of the cubic onto δ from point D, the pole of δ. Hence, the cubic must be embedded in the surface of a quadric cone with vertex D. Further the cubic actually passes through D; for otherwise the projection of the curve onto the plane would be of third order. Similarly, the cubic passes through A, B and C.

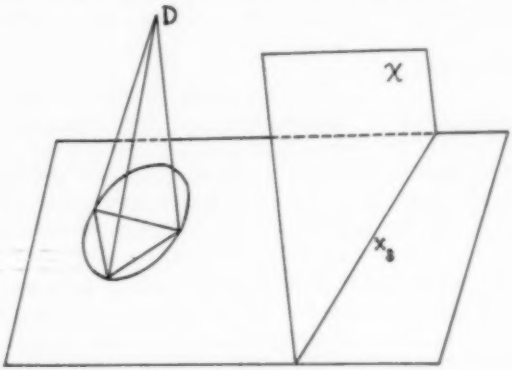

FIGURE 5

(b) (Fig. 6) In a P^3_{41} system, self-dual systems (P^2_{22}) are induced in two faces which have line, say AB, in common. By the nature of such self-dual systems, we see that point X_δ, the projection of X through D onto plane δ, describes a line through C. Likewise, in plane γ, the point X_γ describes a line through D, so that the locus of poles of χ is confined to plane DCA_δ, where A_δ is the companion of $X'_\delta = \chi \cdot AB$ in the involution on line AB. Further, the projection of the locus through B onto β is a conic; and the projection through A onto α is also a conic. Hence, the locus of X is a conic through C, D, and A_δ, where C, D, and A_δ form the vertices of a self-polar tetrahedron for the entire pencil.

(c) (Fig. 7) In a P^3_{42} system, self-dual systems (P^2_{22}) are induced in all the faces of $ABCD$. Let $\chi \cdot AB = X'_\delta$ and $\chi \cdot CD = X'_\alpha$. Let A_α be the companion

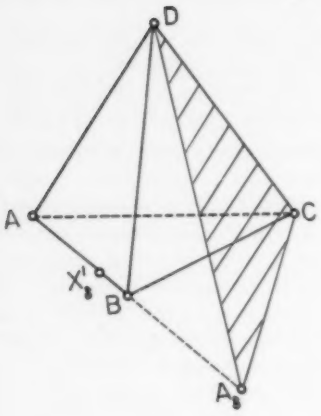

FIGURE 6

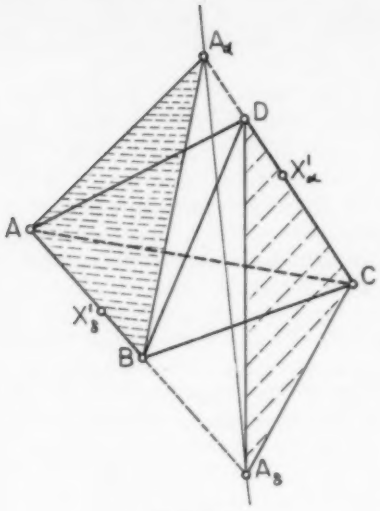

FIGURE 7

of X'_α in the involution on AB, and let A_α be the companion of X'_α in the involution on CD. The projection of X onto δ is the line CA_β; onto plane γ, the line DA_β; and onto plane β, the line AA_α. Therefore, the locus of X must be on planes CDA_β and ABA_α. The locus is, in fact, the meet of these planes, that is, the line $A_\alpha A_\beta$. The points A_α, A_β, X'_α, and X'_β form one of the ∞^2 tetrahedra which are self-polar for the entire pencil.

From previous discussion we know that the polar plane of a fixed point P varies in an axial pencil. We call the axis of the pencil the *axis of point* P and denote it by l_P. It might be noted that there are ∞^3 axes l_P since there are ∞^3 points in S_3. But there are ∞^4 lines in S_3, so we may conclude that not every line is an axis of a point. The following two theorems show that axes of certain related points play interesting roles.

THEOREM 3.8. *The ∞^2 axes l_T of the points T of a fixed plane π are the ∞^2 chords of the twisted cubic of poles of π in a $P^3{}_{40}$ system.*

Proof. Any point T of π can serve as one of the points whose polar planes provide an axial pencil for the generation of the twisted cubic. By well-known theorems on such generation (**5**, p. 170), we know that the axis l_T is a chord of the twisted cubic. Since this is so for all points of π, our theorem is established.

THEOREM 3.9. *If l is a fixed line, the polar lines l'_t of l with respect to the pencil of polarities, and the axes l_P of the different points P of l form the two systems of generators of a quadric, which (in general) passes through the vertices of the self-polar tetrahedron.*

Proof. Take two points, Q and R, arbitrarily on l. Their polar planes, with respect to a polarity of the pencil, intersect in the polar line l'_1, with respect to this polarity. As the polarity varies in the pencil the polar planes of Q and R describe projective pencils with axes l_Q and l_R. The lines of intersection l'_t of corresponding planes describe a regulus, and the axes l_Q and l_R belong to the conjugate regulus **(10**, p. 299**)**.

Now we show that the regulus passes through points A, B, C, D. Let $l\cdot\alpha = A_l$ and $l\cdot\delta = D_l$. The polar of D_l in plane δ varies in a pencil about a point, say D', as the polarity varies; and, in every polarity, the polar planes of A_l pass through A. Therefore, in plane δ, the points of the regulus belong to a conic, which can be thought of as the intersection of corresponding lines of two projective pencils of lines through D' and A. Hence, A, and similarly B, C, and D lie on the regulus.

This construction in plane δ breaks down in the P^2_{42} case, where the polarity induced in one of the faces, say δ, is the same for the entire pencil. In this case, the regulus degenerates.

The first part of the proof meets with a special case in the event that l itself is an axis of a point, say P. In this case the regulus degenerates into a cone with vertex P.

4. Tetrahedral pencils of quadrics.

THEOREM 4.1. *Every pencil of polarities determines a pencil of quadrics.*

Proof. Consider the polarities $(ABCD)(P\pi)$, with π varying through line $p' = l_P$. Since one possible position for π is $P_{p'}$, one of the polarities determines a quadric that touches $P_{p'}$ at P. In fact, every point X lies on such a quadric (touching Xx' at X).

Suppose two of the quadrics have a point R in common. Then all the quadrics have R in common.

COROLLARY. *There is only one quadric through any point which does not lie on the intersection of two.*

The points which are common to all the quadrics of the pencil are simply the points which make up the curve of intersection of the "two." This curve we call Γ, *the base curve*, of the pencil of quadrics. Since two members of the pencil meet a general plane in a conic, and two conics intersect in four points (in general), the base curve is a quartic. It is clear that a pencil of quadrics is cut by a plane in a pencil of conics.

Now if R (on Γ) is on AB, then we are led to one of our cases P^2_{41}, P^2_{42}, or P^2_{43}. In the simplest case, we have self-dual systems determined in δ and γ with lines DR and CR as common tangents. Therefore, plane DCR is a common tangent plane (at R) for all quadrics of the pencil. Further, let Q be on AB such that $H(AB, QR)$. Then DCR is also a common tangent plane (at Q). So, *we have a P^2_{41} system yielding a double-contact of pencil quadrics.*

Suppose in addition to having R (and hence Q) in common, the quadrics also have a point T in common. Three significant cases arise:

(a) T does not lie on any of the faces of $ABCD$. Since plane QRT cuts each of the quadrics in a conic, and since a unique conic is determined by the points Q, R, T (we have the tangents at Q and R), every member of the pencil passes through this conic.

Let plane η be the harmonic conjugate of $ABT(=QRT)$ with respect to γ and δ. Then by well-known properties of quadrics, there is a point $U = \eta \cdot CT$ such that U is also a common point of all the quadrics. Therefore, there is another conic determined by Q, R, U (and the tangents at Q and R), which is common to all the quadrics of the pencil.

Conclusion. This is our $P^3{}_{41}$ case. Γ breaks up into two conics which intersect in R and Q. The two conics lie in planes which are harmonic conjugates with respect to γ and δ.

(b) T is on CD. U, the harmonic conjugate of T with respect to C and D, is also a point common to all the quadrics. Now QT and RT touch all the quadrics at T; QU and RU touch all the quadrics at U. Since Q and R are also on the quadrics, we have lines QT, RT, QU, and RU as common generators for all the members of the pencil.

Conclusion. In this case, $P^3{}_{42}$, Γ is the skew quadrilateral $QTRU$.

(c) T is in plane δ. Again we have the intersection of δ with the pencil of quadrics being a single conic through Q, R, and T. That is, the plane δ has the same polarity induced in it for all members of the pencil. This corresponds to the $P^3{}_{43}$ system. The infinity of lines joining D to the conic in δ are tangents to all the quadrics. That is, the pencil has ring contact on this conic.

Conclusion. A $P^3{}_{43}$ system yields a pencil of quadrics which have ring-contact in plane δ. Γ consists of the conic in δ counted twice. At each point of this conic, the quadrics of the pencil touch.

Another way of arriving at this type of system is to suppose simply that two quadrics of the pencil have a point R in common, with R lying in δ but not on any edge of $ABCD$. Then T, the harmonic conjugate of R with respect to C and $CR \cdot AB$, is also common to the system. Also U, the harmonic conjugate of T with respect to B and $BT \cdot AC$, is common; and V, the harmonic conjugate of U with respect to C and $CU \cdot AB$, is common; and W, the harmonic conjugate of T with respect to A and $BC \cdot AT$, is common. These points, R, T, U, V, W determine a conic which is the intersection of δ with all the quadrics of the pencil.

Suppose that we now begin again with the assumption that two (hence all) of the quadrics have in common a point Q, which does *not* lie on any face of $ABCD$. Then R, the harmonic conjugate of Q with respect to C and $CQ \cdot \gamma$, is common to all the quadrics; point S, the harmonic conjugate of Q with respect to D and $DQ \cdot \delta$, is common to all the quadrics; finally T, the harmonic conjugate of R with respect to D and $DR \cdot \delta$, is common to all the quadrics. Now

Q, R, S, T are coplanar, so if we assume that the quadrics have another point in common in plane $QRST$, we arrive at a previously discussed case, namely P^3_{41}. Therefore, we assume that the quadrics pass through a point U not in plane $QRST$. By the same method used in finding R, S, T, we find V, W, Y coplanar with U, such that V, W, Y are also common to all the quadrics. Hence, we have eight points which determine a quartic, the curve of intersection of the quadrics. This method of finding Γ breaks down if the eight points are associated; for then, any quartic through seven of the points passes through the eighth. In this event we can, by the above method, find many more points belonging to Γ. Since the "eighth associated point" is unique, we have the opportunity of choosing the eighth point (which finally determines Γ) so that it is not associated.

It can be noted that Γ cannot pass through C or D, for these points lie in plane $QRST$. Likewise, the quartic cannot pass through A or B.

Conclusion. This, the most general case, is our P^3_{40}. It yields a pencil of quadrics whose intersection Γ is a quartic which has no multiple points and does not pass through any of the vertices of the fixed self polar tetrahedron.

SUMMARY. *The investigation of tetrahedral pencils of polarities has shown that there are four different types, each yielding a distinct pencil of quadrics. Thus, four of the thirteen non-degenerate pencils of quadrics are reached in this manner. The corresponding Segre symbols* (**6**, pp. 304–308; **9**, pp. 190–195) *are as follows:*

$$P^3_{40} - [1111],\ P^3_{41} - [(11)11],\ P^3_{42} - [(11)(11)],\ P^3_{43} - [(111)1].$$

5. Hexagonal pencils of polarities. Given a self-polar skew hexagon $APBQCR$, a polarity is determined (**7**; pp. 269–288) by the following correspondence of points and planes:

$A \leftrightarrow BQC$,
$P \leftrightarrow QCR$,
$B \leftrightarrow ARC$,
$Q \leftrightarrow PAR$,
$C \leftrightarrow APB$,
$R \leftrightarrow PBQ$,

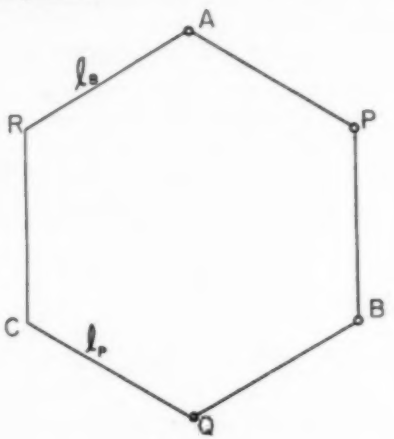

FIGURE 8

DEFINITION. If line RC (Fig. 8) is permitted to vary in a regulus such that R varies along a fixed line through A, while C varies along a fixed line through Q, then we have ∞^1 polarities determined. We call these a *hexagonal pencil of polarities.*

The motivation for this definition follows. The main consideration is to arrive at a definition which generalizes the notion of pencils of polarities as previously defined in S_3. Then suppose we were to let a point, say R, vary. It would have to vary in a twisted cubic, a conic, a line; for these are the loci of poles of a fixed plane PBQ in a pencil of polarities. This sort of definition is, of course, unsuitable. We seek, then, an analogue of the two-dimensional case, and thus consider a line varying. Since its polar line (the opposite side of the hexagon) remains fixed, Theorem 3.9 indicates that our line should vary in a regulus.

Thus we let RC vary in a regulus. Does this mean that lines AR and CQ must also vary? The answer to this question is *no*; for AR and CQ are the axes, l_B and l_P, of the points B and P (respectively), and therefore must be members of the associated regulus. If then AR and CQ were variable, different complementary generators would be intersecting in A and in Q. This situation would always lead to a degenerate regulus.

Therefore, in order to characterize a hexagonal pencil of polarities, we must allow line RC to vary in a regulus, with R always on the fixed line $l_B = AR$, and C always on the fixed line $l_P = QC$.

Define $\alpha = BQC$, and let AP meet α in D. Join AQ and DQ (Fig. 9). The polar plane of Q is $\xi = PAR$. Plane ξ cuts α in DE, where $E = CQ \cdot \xi$. This leads to a self-polar tetrahedron $AQDE$, which is fixed for the entire pencil of polarities.

Now for a point not on the tetrahedron and its polar plane. Consider the point $M = PB \cdot ARQ$. Its polar plane is $\mu = RC(BQ \cdot \xi)$. Call F the point determined by $BQ \cdot \xi$. Then $\mu = RCF$. By Theorem 3.9 we know that D also

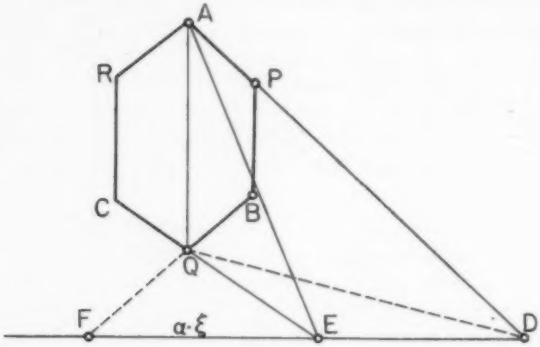

FIGURE 9

lies on the regulus (RC), with the notable exception of the P^3_{43} case. Hence, in general, DE is a generator of the conjugate regulus. Therefore, F in general lies on the quadric.

We wish to show that μ varies in an axial pencil. A projective correspondence exists between the axial pencil CQ and the planes μ, with corresponding planes intersecting in RC. Thus, there is a line l_M — through F and lying in the quadric — which is the axis of the pencil of planes μ.

Now, from Theorem 3.1 we know that the polar planes of any fixed point vary in an axial pencil. In this case, where we have a fixed self-polar tetrahedron, we can show that each axis passes through DE. For the pencil of polarities induces a self-dual system in plane QDE, and the properties of self-dual systems indicate that the polars of any fixed point vary in a pencil whose vertex is always on the same side of the fixed self-polar triangle. Hence, *in general, the axes of the* ∞^2 *points of* S_3 *form a linear complex of lines through* DE.

According as the axis of μ passes through one, two, or three edges of $AQDE$, we get cases P^3_{41}, P^3_{42}, or P^3_{43}. But it appears as though we can never arrive at P^3_{40}. (Note that in the case P^3_{43}, we get one of either R or C fixed for the pencil. The regulus (RC) degenerates into a plane pencil of lines the vertex of which is the fixed point R or C, as the case may be.)

Again, by methods of §4, we can show that a pencil of polarities leads to a system of quadrics, which we call a *pencil of quadrics*.

We know that the hexagon will yield pencils of quadrics P^3_{4k} $(k = 1, 2, 3)$, but the important question to answer is: Do the hexagonal pencils yield any systems of quadrics which do not have fixed self-polar tetrahedra?

In order to find the answer to such a question, we degenerate the self-polar tetrahedron which appears in our hexagonal development. The tetrahedron is $AQDE$. The various ways in which it may degenerate are listed as follows:

5.1 *D lies on plane AQE.*

5.11 $D = A$.

5.111 $\alpha \cdot \xi = AP$, 5.112 $\alpha \cdot \xi = AR$, 5.113 $\alpha \cdot \xi = AB$,
5.114 $\alpha \cdot \xi = AQ$, 5.115 $\alpha \cdot \xi \neq AP, AR, AB, AQ$.

5.12 $D = Q$.

5.121 $\alpha \cdot \xi = AQ$, 5.122 $\alpha \cdot \xi = BQ$,
5.123 $\alpha \cdot \xi = CQ$, 5.124 $\alpha \cdot \xi \neq AQ, BQ, CQ$.

5.13 $D = E$.

5.131 $\alpha \cdot \xi = QC$, 5.132 $\alpha \cdot \xi \neq QC$.

5.14 $D \neq A, Q, E$.

5.2. *E lies on plane ADQ.*

5.21 $E = A$, 5.22 $E = Q$, 5.23 $E = D$, 5.24 $E \neq A, Q, D$.

5.3 *Q lies on plane ADE.*

5.4 *A lies on plane QDE.*

5.41 $A = Q$, 5.42 $A = D$, 5.43 $A = E$, 5.44 $A \neq Q, D, E$.

We proceed to investigate these cases.

5.111 Here A, P, B, Q, C are coplanar (all on α). This case degenerates completely.

5.112 R on α implies that α is the polar plane of P as well as of A. Impossible.

5.113 Since $\alpha \cdot \xi$ is the polar line of AQ, the polar plane of B (on $\alpha \cdot \xi$) always passes through Q as well as A, R, and C. Hence P and B have the same polar plane. Therefore this case degenerates completely.

5.114 (Fig. 10). Since α is a common tangent plane at A for all the quadrics (if the pencil exists), they have contact of order 1 or higher. Actually, they must have four-point contact (third order). For consider the polarities induced in plane BAR. The conics all touch line $AB = a$ at A. Any fixed point on AR has a variable polar through B. This rules out the possibility of constructing a self-polar triangle. Thus two-point contact is ruled out. Since B has a fixed polar line AR, three-point contact is ruled out **(10**, p. 75**)**. Thus the conics must have four-point contact at A; and therefore, the quadrics have four-point contact at A. Similarly for Q.

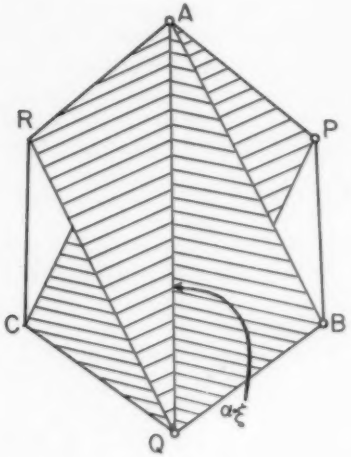

FIGURE 10

Now we look at plane APB. The pencil of conics in this plane must also have four-point contact at A. Again, using a result of Veblen and Young **(10**, p. 75**)**, B has a fixed polar b through A. But $b = ARC \cdot APB$. Therefore C must be fixed on QC. Likewise we can show that R is fixed on AR. Impossible.

5.115 Let $E' = \xi \cdot QB$ (Fig. 11).

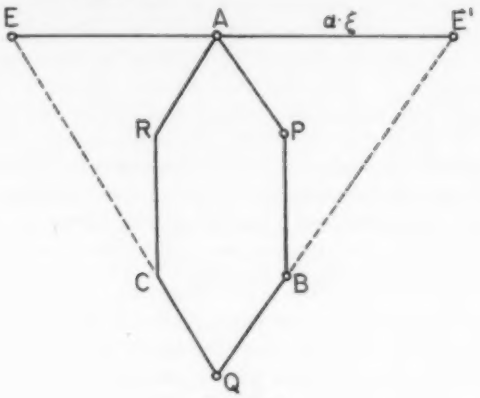

FIGURE 11

5.1151 Suppose there is, in the plane α, a point X, distinct from A, such that X is common to the quadrics of the pencil (if it exists). Since the polarity in α is degenerate, the line x (the polar of X) must be AX.

Since the polar of any point on $\alpha \cdot \xi$ is AQ, X does not lie on AQ. (The situation under discussion may arise when the third generator of the regulus (RC) lies in α. This implies that C remains fixed on the intersection of the third generator and QC.) If we wish C to be our point X, or any point which is self-conjugate for all the polarities, we let C be on AB (Fig. 12). C is on plane APB for every polarity since it is self-conjugate. Let line l be the harmonic conjugate of line AB with respect to AC and $\alpha \cdot \xi$. Then $X_1 = E'C \cdot l$ is also on all the quadrics. Likewise, $X_2 = X_1E \cdot AB$ is on all the quadrics.

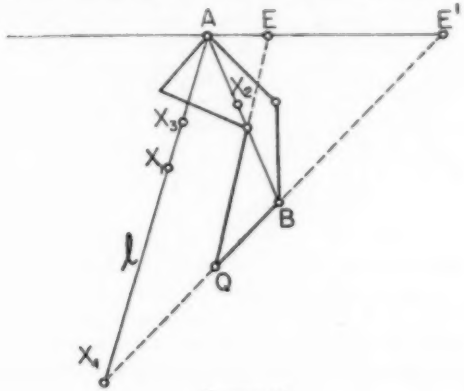

FIGURE 12

Also, $X_3 = E'X_2 \cdot l$. Moreover, the lines AX_2 and AX_3 are common to all the quadrics. α is the common tangent plane at A, APB is the common tangent plane at C, ACR is the common tangent plane at B. A, B, and C are, therefore, double points of the curve of intersection Γ. Actually, the entire line AB is a line of double points.

Now $X_4 = l \cdot BQ$ has as its polar the plane ARX_4, and $X_5 = l \cdot QC$ has APX_5 as its polar plane. Further, these planes are common tangent planes for the pencil of quadrics at points X_4 and X_5, respectively. Hence, line l is also a line consisting of double points of Γ.

Conclusion. We have arrived at a pencil of quadrics whose curve of intersection Γ is a pair of intersecting lines counted twice. All of the quadrics touch at each point of the two lines. We call this a P^3_{61} system.

5.1152 Suppose X is on Γ (that is, is common to all the quadrics) and on ξ, and that X lies on neither line AR nor line AP (Fig. 13). Let $U = EX \cdot AP$. Call X_1 the harmonic conjugate of X with respect to E and U. Let $V = AR \cdot EX_1$. Call X_2 the harmonic conjugate of X_1 with respect to E and V. X_1 and X_2 are points of Γ, too. A unique conic $\mathscr{C}$ is determined in plane ξ by X, X_1, X_2 and A (the tangent at A is $\alpha \cdot \xi$). Hence, the polarity in plane ξ is the same for all the polarities of the pencil. Further, the lines joining Q to the conic $\mathscr{C}$ are tangents to all the quadrics. Therefore this system (if it exists) is a ring-contact system, the same as P^3_{43}.

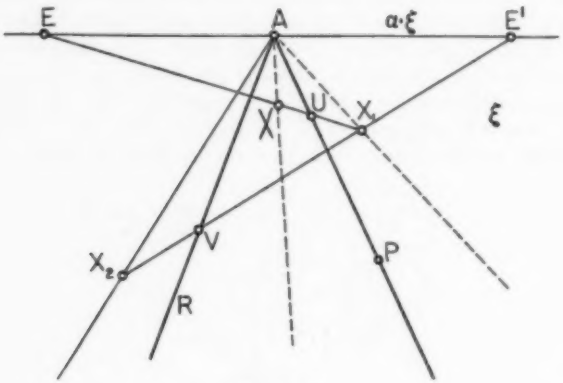

FIGURE 13

5.1153 We now allow a point of Γ to be on AP or AR. By assuming X on Γ to be on AP and taking harmonic conjugates as done above we get two lines of points which belong to Γ. They are all double points because the lines from Q to these lines are all tangents. Hence, all the quadrics touch on our two lines of Γ. *This is a ring-contact system of quadrics touching at a degenerate conic. It reduces to our P^3_{61} case.*

5.1154 Suppose X is on Γ, not on ξ, not on AQP, and not on AQR. For all the polarities of the pencil the polar plane of E' is $\eta' = AQR$ and the polar plane of E is $\eta = AQP$. Let X_1 be such that $H(E(EX\cdot\eta), XX_1)$. Then X_1 is on Γ. Let X_2 be such that $H(E'(E'X\cdot\eta'), XX_2)$. The point A (with its tangent $\alpha\cdot\xi$), X, X_1, X_2 determine a conic $\mathscr{C}_1$ in plane $EE'X$. $\mathscr{C}_1$ is part of Γ. (Additional points of $\mathscr{C}_1$ may be found by repeating above method.) Consider plane θ, the harmonic conjugate of $EE'X$ with respect to α and ξ. The projection of $\mathscr{C}_1$ on θ through point Q is a conic $\mathscr{C}_2$, which also belongs to Γ. *Hence* Γ *consists of two conics which have* $\alpha\cdot\xi$ *as a common tangent at point* A. *We call this a* $P^1{}_{62}$ *system.*

5.1155 If we consider the previous case without the assumption that X is not on η and not on η', we are led to a degenerate conic, consisting of two lines meeting in A, in plane $EE'X$; and in plane θ we also have such a conic. Γ here would consist of four lines all intersecting in A. This is not possible as the intersection of two non-degenerate quadrics. Hence this case does not exist.

5.121 (Fig. 14) $D = Q$ implies that A, P and Q are collinear and $\alpha\cdot\xi = AP$. Therefore the polar line of $\alpha\cdot\xi$ is QC. But $\alpha\cdot\xi$ is its own polar line. Therefore $QC = \alpha\cdot\xi$. For every polarity of the pencil, the polar plane of B is $ARC = \xi$. Hence B and Q always have the same polar plane, which is impossible.

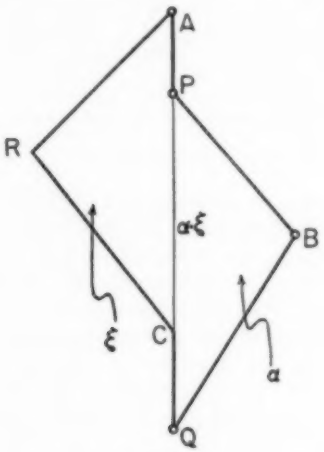

FIGURE 14

5.122 $\alpha\cdot\xi = QB$ is the polar line of AR and AQ. Therefore $AR = AQ$. This case degenerates with A, R, P, B, Q all on ξ, and points B and Q having the same polar plane.

5.123 $\alpha\cdot\xi$ is the polar line of AQ; hence $CQ = \alpha\cdot\xi$ has $AP = AQ$ as its polar line. A, R, P, C, Q are all on ξ. This case degenerates.

5.124 $\alpha\cdot\xi$ is not any of AQ, BP, QB, CQ. The polar line of AQ is $\alpha\cdot\xi$. But $AQ = AP$ (since $D = AP\cdot\alpha$), and the polar line of AP is CQ. Therefore, $\alpha\cdot\xi = CQ$. This contradicts the hypothesis of this case.

5.131 QC is the polar line of AP, and $\alpha\cdot\xi$ is the polar line of AQ. Therefore $AQ = AP$ and $Q = D = E$. This reduces to case 5.123.

5.132 In this case we can find a fixed self-polar tetrahedron $AD'E'Q$, where $E' = BQ\cdot\xi$ and $D' = AR\cdot\alpha$. This reduces to previously discussed cases (Fig. 15).

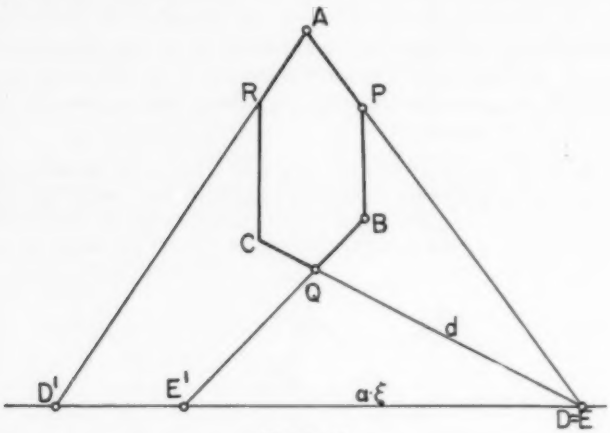

FIGURE 15

However, we may attempt to degenerate $AD'E'Q$ by allowing $D' = E'$. The polar plane of D is APQ for every polarity. Therefore, D is self-conjugate in every polarity. Similarly, the polar plane of D' is AQB and D' is self-conjugate in every polarity. If we now examine the polarity induced in α, we see that $D \leftrightarrow QD$ and $D' \leftrightarrow QD'$. Hence $Q \leftrightarrow \alpha\cdot\xi$. Hence, we have a self-dual system in α. The self-polar triangles are formed by Q and any pair of the involution $(D'D')(DD)$ on $\alpha\cdot\xi$. We can find infinitely many self-polar tetrahedra which are fixed for the pencil by taking A, Q, and a pair of the indicated involution on $\alpha\cdot\xi$. These cases have already been discussed in §3 and §4.

5.14 D lies on AQE. $D \neq A, Q, E$. $QC\cdot\xi = E$ and D is on $\alpha\cdot\xi$. But D is on AQE; hence D is on QE. Therefore $D = E$, which contradicts the hypothesis.

5.21 (Fig. 16). $E = A$ implies A is on QC. The polar plane of P is RCQ, and the polar plane of B is $ARC = RCQ$. Two distinct points have the same polar plane which is impossible.

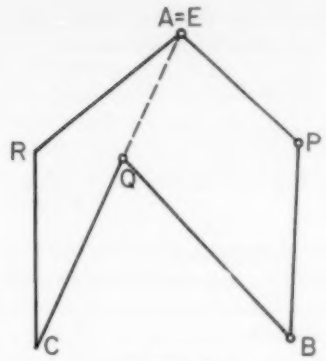

FIGURE 16

5.22 This is the same case as 5.11 with the letters D and E interchanged and the letters A and Q interchanged.

5.23 Although tetrahedron $AQED$ breaks down, we can find another self-polar tetrahedron $AQD'E'$. We can proceed as in 5.132 with the same results.

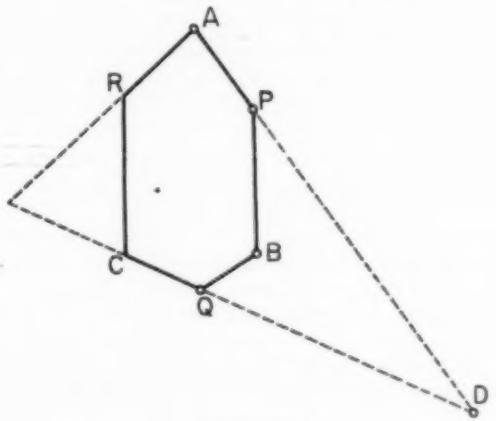

FIGURE 17

5.24 (Fig. 17) E on plane ADQ implies that D is on line QC. Hence, AP and QC are coplanar. $E \neq D$, and ED is $\alpha \cdot \xi = QC$. A, P, Q, C, R are all on ξ. This case is degenerate.

5.3 Q lies on ADE. This is essentially the same as 5.1 and 5.2 where E lies on ADQ, except in the case where $Q = A$. But in this case $\alpha = \xi$, and all the points are coplanar.

5.41 A lies on QDE and $A = Q$. This degenerates; for $A = Q$ implies $\alpha = \xi$, and this implies that all the points are coplanar.

5.42 This case was treated in 5.11.

5.43 This case was treated in 5.21.

5.44 $QDE = \alpha$. Hence, A is on α. $AP \cdot \alpha = D$ implies $A = D$, which case has already been treated.

SUMMARY. *The development of hexagonal pencils of polarities has shown them to be not quite as good as a generalization as pentagonal pencils were in S_2. The hexagonal pencils have yielded two pencils of quadrics which tetrahedral pencils failed to produce. However, one of the systems of quadrics (P^3_{40}) which arose from the tetrahedral development cannot be reached as a system of the hexagonal type.*

Below is a listing of all the systems of quadrics by means of their Segre symbols matched with their pencils of polarities symbols when such exist.

$[1\,1\,1\,1] - P^3_{40}$	$[(2\,1)1] - P^3_{62}$
$[(1\,1)\,1\,1] - P^3_{41}$	$[(2\,1\,1)] - P^3_{61}$
$[(1\,1)(1\,1)] - P^3_{42}$	$[2\,2]$
$[(1\,1\,1)\,1] - P^3_{43}$	$[(2\,2)]$
$[2\,1\,1]$	$[3\,1]$
$[2\,(1\,1)]$	$[(3\,1)]$
	$[4]$

6. Pencils of polarities in S_n. A symmetric (i.e., non-null) polarity in S_n is determined by a self-polar simplex $\Sigma_n = A_1A_2 \ldots A_{n+1}$ and a pair of corresponding elements: point P and the $(n-1)$-flat (prime) π. The notation for this polarity is the usual $(A_1A_2 \ldots A_{n+1})(P\pi)$.

In general, we shall denote the prime $A_1A_2 \ldots A_{i-1}A_{i+1} \ldots A_{n+1}$ by the symbol α_i; α_i is then the polar prime of A_i.

DEFINITION. If π varies in a pencil about the $(n-2)$-flat p, we have ∞^1 polarities $(A_1A_2 \ldots A_{n+1})(P\pi)$ determined. We call these a *pencil of polarities in S_n*.

THEOREM 6.1. *The polar prime χ of any fixed point X, with respect to a pencil of polarities, varies in a pencil about an $(n-2)$-flat.*

Before embarking on a proof we must consider the problem of constructing **(4)** the polar prime of an arbitrary point X. Such a construction can be carried out in S_2 and S_3. We assume that it can also be done in S_{n-1}. Let

$$A_iP \cdot \alpha_i = P_i,\ A_iX \cdot \alpha_i = X_i,\ \pi \cdot \alpha_i = p_i, \qquad i = 1, 2, \ldots, n$$

A_1P is the polar line of the $(n-2)$-flat p_1. Every point on A_1P is conjugate to p_1. In particular, P_1 is such a point. Hence, we have a polarity induced in the prime α_1. By the induction hypothesis, we can find the polar $(n-2)$-flat of point X_1. Call it x_1. We repeat this construction in prime α_2 arriving at an $(n-2)$-flat x_2 which meets x_1. χ, the polar prime of X, must pass through x_1 and x_2, for X is conjugate to every point on x_1 and to every point on x_2. Since these two $(n-2)$-flats meet, they determine a prime which is in fact χ, the polar prime of point X.

Proof of Theorem 6.1. We proceed by induction. The theorem is true for $n = 2, 3$. Suppose it is also true for S_{n-1}.

Let π vary about p. Since p cannot lie in more than $n-1$ of the $n+1$ faces of Σ_n, we suppose that α_1 and α_2 are two in which p does not lie. Then $p \neq p_1 = \pi \cdot \alpha_1$; p_1 varies in a pencil about an $(n-3)$-flat in face α_1. By the induction hypothesis, x_1 likewise varies in a pencil (in α_1) about an $(n-3)$-flat x'_1. Similarly, $x_2 = \chi \cdot \alpha_2$ varies in a pencil (in α_2) about an $(n-3)$-flat which we call x'_2. In each polarity of the pencil we have χ determined by $x_1 \cdot x_2$. χ must pass through $x'_1 \cdot x'_2$ in each of the polarities. Hence, the polar prime χ of point X varies in a pencil about $x'_1 \cdot x'_2$.

In case $x'_1 = x'_2$, then we can take $x'_2, x'_4, \ldots, x'_{n+1}$. They cannot all be equal.

In S_n there are for consideration the cases $P^n_{n+1,k}$ where $k = 0, 1, 2, \ldots, \frac{1}{2}n(n-1)$ according as p meets none, $1, 2, \ldots, \frac{1}{2}n(n-1)$ edges of Σ_n. p passes through the maximum number of edges when it lies entirely in a face, thus meeting all of the $\frac{1}{2}n(n-1)$ edges of Σ_n in the face. It is clear that this is not a complete classification for there may be several distinct pencils in S_n with the same k when $n > 3$; e.g., in P^4_{53}, p meets three edges of Σ_n but different cases occur when the three edges are coplanar and not.

The $P^n_{n+1,1}$ system: In this case we suppose that p meets only one edge, say $A_1A_2 = a$ of Σ_n. Two-dimensional self-dual systems are induced in every plane of Σ_n which is incident with A_1A_2. Hence, the involution of conjugate points on A_1A_2 is the same for all the polarities. Therefore, any pair of the involution on A_1A_2 together with the points $A_3, A_4, \ldots, A_{n+1}$ form a fixed self-polar n-simplex for the pencil of polarities. Hence

THEOREM 6.2. *A $P^n_{n+1,1}$ system has ∞^1 self-polar n-simplexes, each one of which serves for the entire pencil.*

Reasoning analogous to that of Theorem 3.2 gives us

THEOREM 6.3. *The product of two polarities belonging to the same $P^n_{n+1,1}$ system is a general axial homography (a collineation which leaves invariant every point on a line and every prime on an $(n-2)$-flat).*

Proceeding similarly, we have

THEOREM 6.4.

(a) *The $P^n_{n+1,k}$ system has ∞^k self-polar n-simplexes, each of which serves for the entire pencil.*

(b) *If two polarities belong to the same $P^n_{n+1,k}$ system ($k = 2, 3, 4, \ldots, \frac{1}{2}n(n-1) - 1$), their product is a k-axial homography (a collineation having k skew lines which are pointwise invariant).*

The $P^n_{n+1,\frac{1}{2}n(n-1)}$ system is of particular interest since it is the direct generalization of the self-dual system. p now lies entirely in a face, say $\alpha_1 = A_2A_3 \ldots A_{n+1}$, of Σ_n. The polarity induced in this face is the same for each member of the pencil. The polarity is always $(A_2A_3 \ldots A_{n+1})(P_1p)$. Hence the polar primes of point X (for all members of the pencil) pass through x_1. The involutions induced on all the edges in α_1 are the same for every polarity in the pencil. By taking any combination of pairs of the involutions on these edges together with the point A, *we can arrive at $\infty^{\frac{1}{2}n(n-1)}$ self-polar simplexes.* Hence,

THEOREM 6.5. *A $P^n_{n+1,\frac{1}{2}n(n-1)}$ system admits a prime α_1, upon which the induced polarity is the same for all members of the pencil. The polars of any point P pass through a fixed $(n-2)$-flat p on prime α_1, and the poles of any prime π form a range of points on a fixed line A_1P_1, where A_1 is the pole of α_1 for all the polarities.*

THEOREM 6.6. *If two polarities belong to the same $P^n_{n+1,\frac{1}{2}n(n-1)}$ system, their product is an homology.*

Remark. We speak of an homology in S_n when the collineation leaves invariant every prime through a point and every point on a prime.

Proof. We suppose that p, the axis of π, lies in α_1. From the construction of pole and polar prime, we know that every prime through A_1 is fixed under the product of two polarities, and so is every pencil of primes whose axis lies in α_1.

Consider any point X in α_1. Its polar prime χ intersects α_1 in the $(n-2)$-flat x_1. If we operate with another member of the $P^n_{n+1,\frac{1}{2}n(n-1)}$ system, we find that $x_1 \to X$ because the polarity induced in α_1 is the same for all members of the pencil of polarities. Hence, $X \to X$ under the product, which shows that every point of prime α_1 is invariant.

Now consider an arbitrary prime ρ through A_1. Let $\rho \cdot \alpha_1 = r_1$. Let R_1 be the pole of r_1 in plane α_1. Then R, the pole of prime ρ, is on the line A_1R_1. Then operate on R with another member of the system. The construction tells us that R maps into a prime through r_1. Since every line through A_1 is invariant, we have the prime r_1A_1 invariant. Hence, the product is an homology.

THEOREM 6.7. *Every homology can be expressed as the product of two polarities belonging to a* $P^n_{n+1,\frac{1}{2}n(n-1)}$ *system.*

Proof. The homology whose prime of perspectivity is α_1 and whose center is A_1, and with a pair of corresponding points P_1 and P_2 (on a line through A_1) is the product of two polarities $(A_1A_2 \ldots A_{n+1})(P_1\pi)$ and $(A_1A_2 \ldots A_{n+1})(P_2\pi)$, where π is an arbitrary prime.

THEOREM 6.8. *The locus of poles of a fixed prime in general position with respect to a* $P^n_{n+1,0}$ *system is a rational normal n-ic curve.*

Proof. Let χ be a fixed prime. Take n distinct points $Q_1, Q_2, \ldots, Q_n$ on χ. The intersection of the polar primes of these n points is the pole X of χ. The previous theorems indicate that as π varies in the pencil the polar primes of $Q_1, Q_2, \ldots, Q_n$ vary in projective axial pencils. The locus, therefore, is a rational normal *n-ic*.

In $P^n_{n+1,k}$ $(k = 1, 2, 3, \ldots, \frac{1}{2}n(n-1) - 1)$ the rational normal *n-ic* degenerates into an $(n-k)$-ic or lower order curve which is confined to an $(n-k)$-flat. This follows from reasoning precisely the same as used in the proof of Theorem 3.9. The discussion of the $P^n_{n+1,\frac{1}{2}n(n-1)}$ system in Theorem 6.3 indicates that in this case the locus is a line which passes through exactly one vertex A_i of Σ_n, where A_i is the pole of the face of Σ_n with the same induced polarity for all members of the pencil.

Proceeding in a manner analogous to that in §3, we might ask what happens to the ∞^{n-1} axes of l_T of the points T on a fixed prime. The general case is most interesting.

THEOREM 6.9. *The* ∞^{n-1} *axes of* L_T *of the points* T *of a fixed prime* π *are the* ∞^{n-1} *chords of the rational normal n-ic in a* $P^n_{n+1,0}$ *system.*

Proof. Any point T of π can serve as one of the points whose polar prime provides an axial pencil for the generation of the n-ic. By well-known theorems concerned with such a generation, we know that the axis l_T is a chord of the n-ic. Since this is so for all points of π, the theorem is established.

For any given n, a detailed investigation, analogous to that in §4, may be carried out to develop the various pencils of quadrics in S_n. Such a presentation here would be lengthy and, I fear, uninteresting because of its repetitious nature. However, such would not be the case with a development of pencils of polarities and pencils of quadrics by means of a self-polar $(n+2)$-gon. While it is clear that self-polar $(n+2)$-gons always exist in S_n, I confess that I have not discovered any general method of investigation of the pencils of polarities which they yield. The method of attack in S_3 does not readily lend itself to use in higher dimensions; for even in S_4 we would have an enormous number of special cases to treat.

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